MATH70060 – Complex Manifolds – Exercise Sheet 9

Release date: 12 Mar 2025
Submission date: —

Due to lecture scheduling, you *cannot* choose to hand in solutions to this exercise. This exercise sheet does not count towards your module grade.

9.1. Let $X = \mathbb{C}^n$ and
$$\omega = \frac{i}{2} \sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\overline{z_j}$$
be its standard Kähler form. Writing $z_j = x_j + iy_j$, prove that
$$\Delta f = -\sum_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right) f.$$

9.2. Let $X = \mathbb{C}^n/\Lambda$ be a complex torus. Compute $H^{p,q}(X)$ for all $p, q \geq 0$.

9.3. On a compact Kähler manifold, it holds that the map
$$\begin{aligned} \mathcal{H}^{p,q}(X) &\longrightarrow H^{p,q}(X) \\ \alpha &\longmapsto [\alpha] \end{aligned}$$
is a vector space isomorphism for all $p, q \geq 0$. Here $\mathcal{H}^{p,q}(X)$ denote harmonic forms of type (p, q) and $H^{p,q}(X)$ is Definition 4.38. (This will be proved in the lecture, probably on 14 March.)

Use this to prove that on a compact Kähler manifold, every holomorphic p-form $\alpha \in H^0(X, \Omega_X^p)$ is closed.

Note: in an earlier version of this exercise sheet, the word *compact* was missing.

9.4. (Iwasawa manifold) Let $G \subset \mathrm{GL}(3, \mathbb{C})$ be defined as
$$G := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{C} \right\}.$$
Define $\Gamma \subset G$ to be the subgroup of matrices such that $x, y, z \in \mathbb{Z}[i]$.

Show that $X := G/\Gamma$ is a compact complex manifold.

Show that $\mathrm{d}z - x\,\mathrm{d}y$ is a holomorphic $(1, 0)$-form that is not closed.

Use Exercise 9.3 to deduce that X is not Kähler.