MATH70060 – Complex Manifolds – Exercise Sheet 8

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You *can* choose to hand in written solutions to this exercise sheet in hardcopy in class on 12 Mar 2025 and I will correct them. This is optional and does not count towards your module grade.

8.1. Let (X, ω) be a Kähler manifold of complex dimension n. Show that

$$\langle \omega^n, \omega^n \rangle \in C^\infty(X, \mathbb{C})$$

is constant on X.

8.2. Show that ω_{FS} is a Kähler form on $\mathbb{CP}^n$.

8.3. Let X be a complex manifold of complex dimension one. Show that X admits a Kähler form.

8.4. Let X be a compact complex curve (i.e. compact complex manifold of complex dimension 1), and L be a holomorphic line bundle over X. Let s be a holomorphic section of L that has k simple zeros $p_1, \dots, p_k \in X$. In this problem, we will prove that $\int_X c_1(L) = k$.

Recall that $c_1(L) = [\frac{i}{2\pi} F_\nabla]$, where F_∇ is the curvature 2-form of a connection ∇ on L. (For line bundles there is no need to take the trace in the definition of Chern class.)

For each $i = 1, \dots, k$, let U_i be a small open set containing p_i, and z_i be a coordinate on U_i such that the point p_i is given by $z_i = 0$. Let $U_0 = X \setminus \{p_1, \dots, p_k\}$. Note that s gives a frame of L over U_0; choose the connection ∇ with the local formula $\mathrm{d}+A$ for $A = 0$ in this frame.

(a) Given $i \in \{1, \dots, k\}$, write down the expression for ∇ in the local frame $s_i = \frac{1}{z_i} s$ of L over $U_i \cap U_0$. Check that ∇ is not well-defined at $z_i = 0$.

(b) Using smooth cut-off functions (e.g. from a partition of unity), modify the expression for ∇ near each $z_i = 0$, so that the new connection $\widetilde{\nabla}$ is well-defined everywhere.

(c) Calculate the curvature $F_{\widetilde{\nabla}}$ and the integral $\int_X F_{\widetilde{\nabla}}$.
Hint: arrange that $\Theta_{\widetilde{\nabla}}$ vanishes outside small annuli around p_i; use Stokes' theorem.