MATH70060 – Complex Manifolds – Exercise Sheet 7

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Please submit solutions to these exercises on Blackboard. The grade for your submission will count for 5% of your total grade for this course.

7.1. Let $f : X \to Y$ be a smooth map and $E \to Y$ be a complex vector bundle with connection ∇. Then there exists a unique connection $f^*\nabla$ on f^*E satisfying

$$(f^*\nabla)_X(f^*s) = f^*(\nabla_{\mathrm{d}f(X)}s) \quad \text{for} \quad s \in C^\infty(Y, E), X \in \mathfrak{X}(X),$$

where $f^*s := s \circ f \in C^\infty(X, f^*E)$. The connection $f^*\nabla$ is called the *pullback connection*. You need not show that $f^*\nabla$ is a connection or that it is unique.

Let ψ be a local trivialisation of E so that ∇ has the local formula $\nabla = \psi^{-1}(\mathrm{d}+A)\psi$ in this trivialisation. Use this to compute a local formula for $f^*\nabla$. Prove that $F_{f^*\nabla} = f^*(F_\nabla)$. (You can use the local formula for this, but there are other ways.)

7.2. Compute the total Chern class of the bundle $\mathcal{O}(1) \oplus \mathcal{O}(2) \oplus \mathcal{O}(3) \to \mathbb{CP}^3$.

Note: you can use without proof that on any smooth manifold M for line bundles $L \to M$ and $L' \to M$ one has

$$c_1(L \otimes L') = c_1(L) + c_1(L').$$

7.3. Let $E \to X$ be a complex vector bundle with connection ∇. The *dual connection* ∇^* of ∇ is defined via

$$\langle \nabla^* s^*, s \rangle = X(\langle s^*, s \rangle) - \langle s^*, \nabla_X s \rangle \quad \text{for} \quad s \in C^\infty(X, E), s^* \in C^\infty(X, E^*),$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between a vector space and its dual.

Prove that ∇^* is a connection and show that its curvature satisfies

$$\langle F_{\nabla^*} s^*, s \rangle = \langle s^*, -F_\nabla s \rangle.$$

(This can be written compactly as $F_{\nabla^*} = -F_\nabla^T$.)