MATH70060 – Complex Manifolds – Exercise Sheet 6

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You *can* choose to hand in written solutions to this exercise sheet in hardcopy in class on 26 Feb 2025 and I will correct them. This is optional and does not count towards your module grade.

6.1. Consider the trivial vector bundle $E = X \times \mathbb{C}^r$ and let $\nabla = \mathrm{d}{+}A$ be a connection on it. Show that the following three definitions of an induced connection on $\operatorname{End} E$ are equivalent:

(a) Identifying $\operatorname{End} E$, we define a connection ∇' via:

$$(\nabla'\phi)(s) = \nabla(\phi(s)) - \phi(\nabla s) \quad \text{for} \quad \phi \in C^\infty(X, \operatorname{End} E), s \in C^\infty(X, E).$$

(b) On E^*, define a connection ∇'' via

$$(\nabla''\eta)(s) = \mathrm{d}(\eta(s)) - \eta(\nabla s) \quad \text{for} \quad \eta \in C^\infty(X, E^*), s \in C^\infty(X, E).$$

For vector bundles E_1, E_2 with connections ∇_1, ∇_2, defined a connection ∇''' on $E_1 \otimes E_2$ via

$$\nabla'''(s_1 \otimes s_2) = (\nabla_1 s_1) \otimes s_2 + s_1 \otimes (\nabla_2 s_2). \tag{$*$}$$

Using the identification $\operatorname{End} E \cong E \otimes E^*$, the two definitions taken together induce a connection on $\operatorname{End} E$.

(c) Locally, define ∇'''' via

$$\nabla''''\phi = \mathrm{d}\phi + [A, \phi] \quad \text{for} \quad \phi \in C^\infty(X, \mathfrak{gl}(r, \mathbb{C})),$$

where $[\cdot, \cdot]$ denotes taking the commutator on the endomorphism part.

6.2 By using $(*)$ from above repeatedly, one can define a connection on $E^{\otimes k}$. Write down a formula for it.

Show that the same formula is well defined on $\bigwedge^k E$ given as $(\bigwedge^k E)_x = E_x^{\otimes k}/I$, where $I \subset E^{\otimes k}$ is the ideal generated by elements $v \otimes v$ for $v \in E$. Show that this gives rise to the following connection:

$$\nabla(s_1 \wedge \cdots \wedge s_k) = \sum_{i=1}^{k} s_1 \wedge \cdots \wedge \nabla s_i \wedge \cdots \wedge s_k.$$

Show if $\nabla = \mathrm{d}{+}A$ is a connection on $E = X \times \mathbb{C}^r$, then the connection induced on $\det E$ in this way is $\mathrm{d}{+}\operatorname{tr}(A)$.

6.3. Show that a connection ∇ on a Hermitian vector bundle (E, h) is Hermitian if and only if $\nabla h = 0$, where by ∇ we also denote the naturally induced connection on the bundle $(E \otimes \overline{E})^*$.