MATH70060 – Complex Manifolds – Exercise Sheet 5

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You *can* choose to hand in written solutions to this exercise sheet in hardcopy in class on 19 Feb 2025 and I will correct them. This is optional and does not count towards your module grade.

5.1. Let $E \to X$ be a complex vector bundle and $\psi, \psi' : E|_U \to U \times \mathbb{C}^r$ be two local trivialisations such that $\phi \cdot \psi' = \psi$. Write

$$\nabla = \psi^{-1}(d + A)\psi = (\psi')^{-1}(d + A')\psi'.$$

Prove that

$$\mathrm{d}A' + A' \wedge A' = \phi^{-1}(\mathrm{d}A + A \wedge A)\phi.$$

5.2. Let ∇ be a connection on a complex vector bundle $E \to X$. Prove the *Bianchi identity*:

$$\nabla(F_\nabla) = 0 \in C^\infty(X, \Omega^3_{X,\mathbb{C}} \otimes \mathrm{End}(E)).$$

5.3. Let (E, h) be a Hermitian vector bundle on a smooth manifold X. Show that for every $x \in X$ there exists a neighbourhood U of x and a local frame $s_1, \ldots, s_r \in C^\infty(U, E|_U)$ with the property

$$\langle s_i, s_j \rangle = \delta_{ij} \text{ for } i, j \in \{1, \ldots, r\}.$$

5.4. Let ∇ be a connection on a line bundle $L \to X$. Prove the following: we have $F_\nabla = 0$ if and only if there exists a local parallel nowhere zero section around every point.

Hint: the local formula for the curvature of a bundle is $\mathrm{d}A + A \wedge A$, and for line bundles the second summand vanishes (why?). Then use $\mathrm{d}A = 0$ and the Poincaré lemma.

Remark: The corresponding statement for vector bundles of arbitrary rank is true. That is: the curvature is zero if and only if there exists a local parallel frame of the bundle. This is sometimes called *Frobenius integrability*.