MATH70060 – Complex Manifolds – Exercise Sheet 4

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Please submit solutions to these exercises on Blackboard. The grade for your submission will count for 5% of your total grade for this course.

4.1. Let $E \to X$ be a holomorphic vector bundle over a complex manifold. In the lecture, the $\mathbb{C}$-linear operator $\overline{\partial}_E : C^\infty(X, \Omega_X^{p,q}(E)) \to C^\infty(X, \Omega_X^{p,q+1}(E))$ was defined. It was shown that is satisfies the following two properties:

(a) $\overline{\partial}_E s = 0$ for every holomorphic section of E,

(b) for $\alpha \in C^\infty(X, \Omega_X^{p,q})$, $\beta \in C^\infty(X, \Omega_X^{r,s}(E))$ the Leibniz rule

$$\overline{\partial}_E(\alpha \wedge \beta) = (\overline{\partial}\alpha) \wedge \beta + (-1)^{p+q} \alpha \wedge \overline{\partial}_E \beta$$

holds.

Prove that any $\mathbb{C}$-linear operator $C^\infty(X, \Omega_X^{p,q}(E)) \to C^\infty(X, \Omega_X^{p,q+1}(E))$ satisfying these two properties must be equal to $\overline{\partial}_E$.

4.2. Let $E, F \to X$ be holomorphic vector bundles over a complex manifold X. Let $\alpha : E \to F$ be a bundle morphism. Show that

$$\alpha^* : C^\infty(X, \Omega^{0,q}(E)) \to C^\infty(X, \Omega^{0,q}(F))$$
$$\omega \otimes s \mapsto \omega \otimes \alpha(s)$$

is well defined as a map $H^q(X, E) \to H^q(X, F)$.

4.3. On $\mathbb{R}^2$ we write (x, y) for the standard coordinates and define an almost complex structure via

$$J : T(\mathbb{R}^2) \to T(\mathbb{R}^2)$$
$$\frac{\partial}{\partial x} \mapsto e^x \frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial y} \mapsto -e^{-x} \frac{\partial}{\partial x}.$$

Prove that J is integrable.

4.4. Compute the Hodge numbers of $\mathbb{CP}^1$.

Hint: there are different ways to compute this. If you want, you can use without proof that the *Euler sequence* on $\mathbb{CP}^1$ is a short exact sequence, i.e. there are linear maps making the following a short exact sequence:

$$0 \to \Omega_{\mathbb{CP}^1}^{1,0} \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{CP}^1} \to 0.$$