MATH70060 – Complex Manifolds – Exercise Sheet 3

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You *can* choose to hand in written solutions to this exercise sheet in hardcopy in class on 5 Feb 2025 and I will correct them. This is optional and does not count towards your module grade.

3.1. Let X be a complex manifold.

Prove the following: A holomorphic line bundle $L \to X$ is isomorphic to the the trivial line bundle if and only if it admits a section that is nowhere zero.

3.2. Let E and F be vector bundles with trivialisations ψ_i and ϕ_i. Denote their transition functions by g_{ij} and h_{ij}. Define a trivialisation for $E \oplus F$ and E^* and find an expression for their transition functions in terms of g_{ij} and h_{ij}.

Use this to show that the transition functions of $\mathcal{O}(1)$ over $\mathbb{CP}^n$ with respect to the trivialisations of $\mathcal{O}(-1)$ defined in the lecture are

$$\begin{aligned} g_{ij} : U_i \cap U_j &\to \mathrm{GL}(1, \mathbb{C}) \\ ([x_0 : \cdots : x_n]) &\mapsto \frac{x_j}{x_i}. \end{aligned}$$

3.3. Let V be an $\mathbb{R}$-vector space. Show that the following $\mathbb{C}$-vector spaces are isomorphic:

$$\left(\bigwedge_{\mathbb{R}}^k V \right) \otimes_{\mathbb{R}} \mathbb{C} \cong \bigwedge_{\mathbb{C}}^k (V \otimes_{\mathbb{R}} \mathbb{C}).$$

Check that this implies that the following complex vector bundles (over some smooth manifold X) are isomorphic:

$$\Omega_X^{k+1} \otimes_{\mathbb{R}} \mathbb{C} \cong \Omega_{X,\mathbb{C}}^{k+1}.$$

3.4. The goal of this exercise is to prove that holomorphic sections of $\mathcal{O}(1)$ over $\mathbb{CP}^1$ are in one-to-one correspondence with homogeneous degree one polynomials in the variables x_0, x_1. As usual, $\mathbb{CP}^1 = U_0 \cup U_1$ denotes the standard affine cover, and ψ_0, ψ_1 the trivialisations and g_{01} the transition function from exercise 3.2.

(a) Let

$$\begin{aligned} s_0 : U_0 &\to U_0 \times \mathbb{C} & \qquad s_1 : U_1 &\to U_1 \times \mathbb{C} \\ [1 : x_1] &\mapsto ([1 : x_1], f(x_1)), & [x_0 : 1] &\mapsto ([x_0 : 1], h(x_0)) \end{aligned}$$

for some $f, h : \mathbb{C} \to \mathbb{C}$.
Show the following: if there exists $s \in H^0(\mathbb{CP}^1, \mathcal{O}(1))$ such that $\psi_0 \circ s\, |_{U_0} = s_0$ and $\psi_1 \circ s\, |_{U_1} = s_1$, then f and h are holomorphic and $g_{01} s_1\, |_{U_0 \cap U_1} = s_0\, |_{U_1 \cap U_0}$. This last equation implies that $h(x) = x \cdot f(1/x)$ for $x \in \mathbb{C} \setminus \{0\}$. Deduce that f and h are polynomials of degree 1 (not necessarily homogeneous).

(b) Let p be a homogeneous polynomial of degree 1 in x_0, x_1. Denote

$$f : \mathbb{C} \to \mathbb{C} \qquad\qquad h : \mathbb{C} \to \mathbb{C}$$
$$x \mapsto p(1, x), \qquad\qquad x \mapsto p(x, 1).$$

Show the following: there exists $s \in H^0(\mathbb{CP}^1, \mathcal{O}(1))$ such that $\psi_0 \circ s\, |_{U_0} = s_0$ and $\psi_1 \circ s\, |_{U_1} = s_1$, where s_0, s_1 are defined by the equations above.

(c) Use part (b) to define a map from homogeneous polynomials of degree 1 into $H^0(\mathbb{CP}^1, \mathcal{O}(1))$. Use part (a) to show surjectivity of this map. Check that it is also injetive.