MATH70060 – Complex Manifolds – Exercise Sheet 2

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You *can* choose to hand in written solutions to this exercise sheet in hardcopy in class on 29 Jan 2025 and I will correct them. This is optional and does not count towards your module grade.

2.1. Prove that $\mathbb{CP}^n$ and $S^{2n+1}/\mathrm{U}(1)$ are homeomorphic, where

$$S^{2n+1} := \{z = (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : |z| = 1\}$$

and $\mathrm{U}(1) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ acts on $\mathbb{C}^{n+1}$ via $\lambda \cdot (z_1, \dots, z_{n+1}) = (\lambda z_1, \dots, \lambda z_n)$. Conclude that $\mathbb{CP}^n$ is compact.

2.2. Show that there exists no embedding of $\mathbb{CP}^k$ into $\mathbb{C}^n$ for any values of $k, n \geq 1$.

Hint: use that any holomorphic function on a compact manifold is constant.

2.3. For $\tau \in \mathbb{C}$ let $\Lambda(\tau) := \mathrm{span}_{\mathbb{Z}}\{1, \tau\}$. Prove the following:

(a) The complex manifolds $\mathbb{C}/\Lambda(\frac{1}{2}i)$ and $\mathbb{C}/\Lambda(2i)$ are biholomorphic.

(b) The complex manifolds $\mathbb{C}/\Lambda(\exp(\frac{\pi i}{3}))$ and $\mathbb{C}/\Lambda(\exp(\frac{2\pi i}{3}))$ are biholomorphic.
(Note: an earlier version of the exercise sheet contained a typo here. exp was missing.)

2.4.* With the notation $\Lambda(\tau)$ from the previous exercise, make some choice of τ_1 and τ_2 for which $\mathbb{C}/\Lambda(\tau_1)$ and $\mathbb{C}/\Lambda(\tau_2)$ are *not* biholomorphic.

(This is probably a very hard question. I only know how to prove this by using the j-invariant, which is a complicated construction. I would be interested in an example with an easy proof.)

2.5. (a) Let V be a real vector space and $L : V \to L$ be an $\mathbb{R}$-linear map. Let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification and $L_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of L. Show that

$$\det{}_{\mathbb{R}}(L) = \det{}_{\mathbb{C}}(L_{\mathbb{C}}).$$

(b) Let V be a complex vector space and $L : V \to V$ be a $\mathbb{C}$-linear map. Let $V_{\mathbb{R}}$ be the underlying real vector space and $L_{\mathbb{R}}$ the map L viewed as an $\mathbb{R}$-linear map. Show that

$$|\det{}_{\mathbb{C}}(L)|^2 = \det{}_{\mathbb{R}}(L_{\mathbb{R}})$$

(c) Let V be a complex vector space and $L : V \to V$ be a $\mathbb{C}$-linear map. Let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification and $L_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of L. Show that

$$|\det{}_{\mathbb{C}}(L)|^2 = \det{}_{\mathbb{C}}(L_{\mathbb{C}}).$$