MATH70060 – Complex Manifolds – Exercise Sheet 1

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You *can* choose to hand in written solutions to this exercise sheet in hardcopy in class on 22 Jan 2025 and I will correct them. This is optional and does not count towards your module grade.

1.1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$. Show that the following are equivalent:

(a) Under the identification $\mathbb{R}^2 = \mathbb{C}$ the function f is holomorphic.

(b) For all $(x_0, y_0) \in \mathbb{R}^2$ the differential $\mathrm{d}f_{(x_0,y_0)} : \mathbb{R}^2 \to \mathbb{R}^2$ is complex linear if we identify $\mathbb{R}^2 = \mathbb{C}$.

(c) For the *Wirtinger derivatives* $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ we have $\frac{\partial}{\partial \bar{z}} f = 0$.

Notation suggestion: write multiplication by i on $\mathbb{C}$ as $i\cdot$ and the induced map on $\mathbb{R}^2$ as $J : \mathbb{R}^2 \to \mathbb{R}^2$.

1.2. Define a smooth manifold structure on $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ so that $\hat{\mathbb{C}}$ is diffeomorphic to S^2.

1.3. Identify $\mathbb{C} = \mathbb{R}^2$ via $z = x + iy$. Compute

$$\int_{\mathbb{C}} \left|\frac{1}{z}\right| \mathrm{d}x \wedge \mathrm{d}y.$$