Complex Manifolds

Daniel Platt (15th June 2025)
(based on lecture notes by Paolo Cascini and Mykola Matviichuk)

Contents

1 Introduction

A complex manifold is about the same thing as a differentiable manifold, but everywhere you see the word "diffeomorphism" replace it with "holomorphic isomorphism" or "biholomorphism". In this introduction, we will list some examples that will turn out to be complex manifolds later. We will give the rigorous definition later, this is just to get an idea of what spaces this lecture will be about.

Example 1.1. $\mathbb{C}^n$ is a complex manifold. In fact, any open subset in $\mathbb{C}^n$ is a complex manifold.

Example 1.2. The sphere S^2 is homeomorphic to the one-point compactification of the complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. It can be given the structure of a complex manifold. More generally, $\mathbb{CP}^n$ is a complex manifold. Instead of generalising S^2 to $\mathbb{CP}^n$, one can also consider $S^4, S^6, \ldots$ Is is a theorem that S^{2n} is *not* a complex manifold, except for $n = 1$ and $n = 3$. The converse is an open problem: we know that S^2 *is* a complex manifold, but for S^6 it is not known whether it is or isn't. This is called the *Hopf Problem.*

Example 1.3. The torus $T^2 := \mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{C}/\mathbb{Z}^2 \cong S^1 \times S^1$ is a complex manifold. More generally, any $2n$-dimensional lattice $\Lambda \subset \mathbb{C}^n$ defines the $2n$-torus $T^{2n} := \mathbb{C}^n/\Lambda$ which is also a complex manifold. On the other hand, S^{2n} is not a complex manifold for $n \neq 1, 3$. It is unknown if S^6 is a complex manifold.

Example 1.4. Any genus g surface is a complex manifold.

Example 1.5. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Then the graph of f,

$$\Gamma_f = \{(z, f(z))\} \subset \mathbb{C} \times \mathbb{C}$$

is a complex manifold. Given Γ_f we can recover f as follows:

$$f(z) = q\left(p^{-1}(z) \cap \Gamma_f\right)$$

where p, q are the projection onto the first and second coordinate respectively.

More in general, given any complex submanifold $\Gamma \subset \mathbb{C} \times \mathbb{C}$, we can define a “multivalued holomorphic function” by

$$f_\Gamma(z) = q\left(p^{-1}(z) \cap \Gamma\right).$$

In particular, this allows to construct the inverse of a function: let

$$\tau : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$$

be defined by $\tau(z, w) = (w, z)$ and given $f\colon \mathbb{C} \to \mathbb{C}$, let

$$\Gamma_{f^{-1}} = \tau(\Gamma_f).$$

Then $f^{-1} = f_{\Gamma_{f^{-1}}}$ is the inverse of f. For example, $\log(z)$ is the multivalued holomorphic function defined as the inverse of $f(z) = \exp(z)$.

Example 1.6. Generalising the previous example, we can consider holomorphic maps between complex manifolds:

$$f\colon M \to N.$$

For example, given M, what are the automorphism of M:

$$\mathrm{Aut}(M) = \{f\colon M \to M \mid f \text{ is biholomorphic}\}.$$

There are many maps $f\colon \mathbb{C} \to \mathbb{C}$ but the only automorphism of $\mathbb{C}$ are affine linear maps.

The study of automorphisms of a torus $f\colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ has many applications in cryptography.

Example 1.7. Algebraic geometry is the study of the zeroes of polynomials. Given a collection of polynomials, $f_1, \ldots, f_k$ in variables $x_1, \ldots, x_m$, then the set

$$\{(x_1, \ldots, x_m) \in \mathbb{C}^m \mid f_1 = \cdots = f_k = 0\}$$

is called algebraic variety. If it is smooth, then it is also a complex manifold.

If $F_1, \ldots, F_k$ are homogeneous polynomials in variables $x_0, \ldots, x_m$ then the set

$$\{(x_0, \ldots, x_m) \in \mathbb{P}^m \mid F_1 = \cdots = F_k = 0\}$$

is called projective variety. Also in this case, if it is smooth, then it is a complex manifold. Surprisingly, even the converse is true! Namely, any analytic complex submanifold of $\mathbb{P}^n$ is given as the zero locus of homogeneous polynomials. This is *Chow's theorem.*

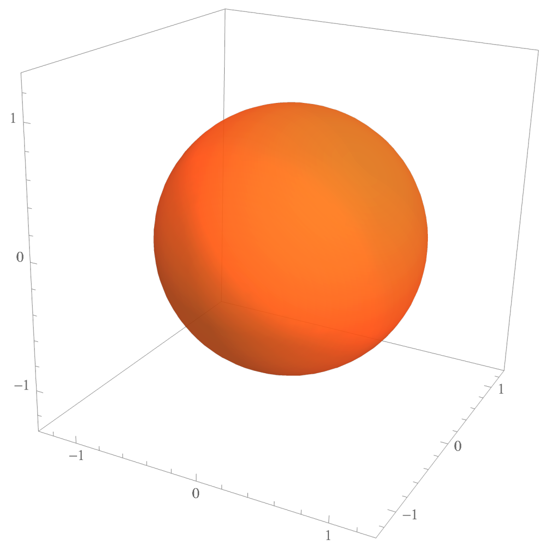

Figure 1: A (very misleading) picture of the variety $\{(x_1, x_2) \in \mathbb{C}^2 : x_1^2 - x_2^2 - 1 = 0\}$. It is misleading, because the picture only shows the real points of the variety, and not those points with imaginary components.

Note that a differentiable manifold X contains many compact submanifolds. For example, through every point $x \in X$, there eists a positive dimensional submanifolds. On the other hand, there are complex manifolds which do not admit any proper complex submanifolds.

Question 1.8. What can we say about the topology of a complex manifold? Is it orientable? What can the fundamental group be? Can we list all the simply connected complex manifolds? What can the (co)homology groups look like?

One way in which we will address this question in this lecture is by introducing *Dolbeault cohomology*, which contains a lot of information of complex manifolds.

Question 1.9. What is the relationship between the complex structure and the Riemannian structure on a manifold?

Manifolds on which both structures are present are called *Kähler manifolds*. In this lecture we will prove a few properties of these manifolds, showing that the complex and Riemannian structures are indeed very strongly related! We may even get as far as to say some things about *Calabi-Yau manifolds*, which are a special class of Kähler manifolds that are interesting in many areas of pure mathematics and physics.

2 Local Theory

2.1 Holomorphic functions in several variables

We will now recall some facts about holomorphic functions in several variables.

Definition 2.1. We denote by $D(z_0, r)$, the complex disc of radius r centred at z_0:

$$D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

The boundary of $D(z, r)$ will be denoted by $\partial D(z_0, r)$.

Definition 2.2. Let $U \subset \mathbb{C}$ be an open subset. Let $f : U \to \mathbb{C}$ be a continuous function. Then f is *holomorphic* on U if for all $z_0 \in U$, the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Theorem 2.3 (Cauchy's integral formula). *Let $U \subset \mathbb{C}$ be an open subset and let $f : U \to \mathbb{C}$ be holomorphic. Let $z_0 \in U$ and assume that $D := D(z_0, r)$ is such that $\overline{D} \subset U$.*
Then for every $\xi \in D$, one has

$$f(\xi) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - \xi} dz.$$

We will prove a more general claim below (Proposition 2.15), which will imply Theorem 2.3.

Definition 2.4. Let $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ and let $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. We will denote by $D(c, r)$ the *polydisc* centred at c with polyradius r, i..e

$$D(c, r) = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j - c_j| < r_j \text{ for all } j\}.$$

Definition 2.5. Let $U \subset \mathbb{C}^n$ be an open set. Let $f : U \to \mathbb{C}$ be a continuous function. We say that f is *holomorphic* if for each $z = (z_1, \ldots, z_n) \in U$ such that $D(z, \epsilon) \subset U$ for some polyradius $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, we have that the function in one variable

$$f(z_1, \ldots, z_{i-1}, \cdot, z_{i+1}, \ldots, z_n) : D(z_i, \epsilon_i) \to \mathbb{C}$$

is holomorphic.

Example 2.6. Any convergent power series in n variables is holomorphic.

We will now see that also the converse is true.

Theorem 2.7 (Cauchy). *Let $U \subset \mathbb{C}^n$ be an open set and let $f : U \to \mathbb{C}$ be holomorphic. Let $z = (z_1, \ldots, z_n) \in U$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n_{>0}$ be such that $\overline{D(z, \epsilon)} \subset U$.*
Then for every $\xi = (\xi_1, \ldots, \xi_n) \in D(z, \epsilon)$, we have

$$f(\xi) = \frac{1}{(2\pi i)^n} \int_{\partial D(z_1, \epsilon_1)} \cdots \int_{\partial D(z_n, \epsilon_n)} \frac{f(z)}{(z_1 - \xi_1) \ldots (z_n - \xi_n)} dz_n \ldots dz_1$$

The Theorem follows by induction on n, by applying Equation 2.3 at each step. It follows:

Corollary 2.8. *Let $U \subset \mathbb{C}^n$ be an open set and let $f : U \to \mathbb{C}$ be holomorphic. Let $z = (z_1, \ldots, z_n) \in U$. Then there exists $D := D(z, \epsilon) \subset U$ for some polyradius $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and a power series*

$$p(\xi) = \sum_{m_1, \ldots, m_n \geq 0} a_{\bar{m}} (\xi_1 - z_1)^{m_1} \ldots (\xi_n - z_n)^{m_n}$$

such that p is convergent on D and $p(\xi) = f(\xi)$ on D.

Proof. We give the proof idea for the case $n = 1$, the general case is done analogously.
By the previous theorem we have that

$$\begin{aligned} f(\xi) &= \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-\xi} \,\mathrm{d}w \\ &= \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)-(\xi-z)} \,\mathrm{d}w \\ &= \frac{1}{2\pi i} \int_{\partial D} \frac{1}{w-z} \cdot \frac{f(w)}{1-\frac{\xi-z}{w-z}} \,\mathrm{d}w \\ &= \frac{1}{2\pi i} \int_{\partial D} \frac{1}{w-z} \sum_{k=0}^{\infty} \left(\frac{\xi-z}{w-z}\right)^k f(w) \,\mathrm{d}w \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} (\xi-z)^k \cdot \int_{\partial D} \frac{f(w)}{(w-z)^{k+1}} \,\mathrm{d}w, \end{aligned}$$

where we used the geometric series in the fourth step. The last line is a power series in $(\xi - z)$ which proves the claim. □

Definition 2.9. Let $U \subset \mathbb{C}^n$ be an open set. A function $f\colon U \to \mathbb{C}^m$ is **holomorphic**, if for each projection $p_i\colon \mathbb{C}^m \to \mathbb{C}$, the function

$$f_i = p_i \circ f\colon U \to \mathbb{C}$$

is holomorphic.

Note that if $f\colon U \to V$ and $g\colon V \to W$ are holomorphic, then the composition $g \circ f\colon U \to W$ is also holomorphic.

Definition 2.10. Let $U \subset \mathbb{C}^n$ be an open set. A holomorphic function $f\colon U \to \mathbb{C}^m$ is **biholomorphic** at a point $z \in U$ if there exists a neighbourhood $z \in V \subset U$ such that $f\colon V \to f(V)$ is bijective and $f^{-1}\colon f(V) \to V$ is holomorphic.
We say that f if biholomorphic if it is a bijection and biholomorphic at all points $z \in U$.

Note that, in the assumptions above, $f(V)$ is automatically an open set of $\mathbb{C}^m$.

Example 2.11. Let A be an invertible $n \times n$ complex matrix. Then A defines a biholomorphism $f\colon \mathbb{C}^n \to \mathbb{C}^n$.

Example 2.12. Let $f(z)\colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ defined by $f(z) = z^2$. Then f is a biholomorphism at each point, but it is not a biholomorphism.

Identifying $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, it follows that any holomorphic function is real analytic and, hence, C^∞. Thus, if $f\colon U \to V$ is a biholomorpihsm, it is also a diffemorphism and a homeomorphism.

Theorem 2.13 (Hartog's theorem). *Let $n \geq 2$ and let $R = (R_1, \ldots, R_n)$ and $r = (r_1, \ldots, r_n)$ such that $R_i > r_i > 0$ for $i = 1, \ldots, n$. Let*

$$U = D(0, R) \setminus D(0, r) \subset \mathbb{C}^n.$$

Then any holomorphic function on U extends to a holomorphic function on $D(0, R)$.

Note that this is false if $n = 1$. Indeed, it is enough to consider the function $f(z) = \frac{1}{z}$.
We omit the proof here. It can be found in [7, Proposition 1.1.4].

2.2 Cauchy formula in one variable

Identify

$$\begin{aligned} z : \mathbb{R}^2 &\to \mathbb{C} \\ (x, y) &\mapsto x + iy. \end{aligned}$$

This *implies*

$$\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y \text{ and } \mathrm{d}\overline{z} = \mathrm{d}x - i\,\mathrm{d}y$$

and

$$\frac{i}{2}\,\mathrm{d}z \wedge \mathrm{d}\overline{z} = \mathrm{d}x \wedge \mathrm{d}y,$$

which is the Lebesgue measure on $\mathbb{R}^2$. For now, the symbols $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ have no meaning, but we can *define*

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \text{ and } \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

These are called *Wirtinger derivative.*
Let $U \subset \mathbb{C}$ open and $f : U \to \mathbb{C}$ s.t. it is smooth if viewed as a map $f : U \to \mathbb{R}^2$.

Exercise 2.14. *Show that $f : U \to \mathbb{C}$ is holomorphic if and only if $\frac{\partial f}{\partial \overline{z}} = 0$.*

Proposition 2.15 (Cauchy's integral formula for C^∞-functions). *Let $r > 0$ and let $D := D(0, r)$. Let $f : U \to \mathbb{C}$ be a C^∞-function, where $U \subset \mathbb{C}^n$ is an open set containing $\overline{D}$.*
Then for every $\xi \in D$, we have

$$f(\xi) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z-\xi} dz + \frac{1}{2\pi i}\int_D \frac{\partial f}{\partial \bar{z}} \cdot \frac{dz \wedge d\bar{z}}{z - \xi}.$$

Proof. We will assume, for simplicity, that $\xi = 0$.

Exercise 2.16. *$g(z) = \frac{1}{z}$ is absolutely integrable over D with respect to the measure $dz \wedge d\bar{z}$.*

Thanks to this exercise, we have

$$\int_D \frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z} = \lim_{\epsilon \to 0} \int_{D - D(0,\epsilon)} \frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z}.$$

Away from zero, we have

$$\frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z} = -d\left(f(z)\frac{dz}{z}\right).$$

(Here, d denotes the exterior derivative of a 1-form.) Thus, by Stokes theorem, we have

$$\lim_{\epsilon \to 0} \int_{D - D(0,\epsilon)} \frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z} = -\int_{\partial D} f(z)\frac{dz}{z} + \lim_{\epsilon \to 0}\int_{\partial D(0,\epsilon)} f(z)\frac{dz}{z}.$$

Since

$$\lim_{\epsilon \to 0}\int_{\partial D(0,\epsilon)} f(z)\frac{dz}{z} = 2\pi i f(0),$$

the result follows. □

Note that if f is holomorphic, then $\frac{\partial f}{\partial \bar{z}} = 0$ and thus, we recover Equation 2.3.

2.3 Implicit function theorem

Definition 2.17. Let $U \subset \mathbb{C}^n$ be an open set. Given a holomorphic function $f: U \to \mathbb{C}^m$, we define the **holomorphic Jacobian** of f at $z \in U$, to be the matrix J_f defined by

$$J_f(z) = \left(\frac{\partial f_i}{\partial z_j}(z)\right)_{i,j}$$

where $f_i = p_i \circ f$ and $p_i: \mathbb{C}^m \to \mathbb{C}$ is the i-th projection.

Theorem 2.18 (Implicit function theorem). *Let $U \subset \mathbb{C}^{n+k}$ be an open set and let $f: U \to \mathbb{C}^n$ be a holomorphic function. Let $z \in U$ be a point where the Jacobian J_f has rank n. After renumbering the coordinates of $\mathbb{C}^{n+k}$, if necessary, let us assume that the square submatrix*

$$J_f(z) = \left(\frac{\partial f_i}{\partial z_j}(z)\right)_{1 \le i,j \le n}$$

is non-singular.
Then there are open subsets $U_1 \subset \mathbb{C}^n$, $U_2 \subset \mathbb{C}^k$ and a holomorphic function $g : U_2 \to U_1$ such that $U_1 \times U_2 \subset U$ and the level set

$$\left\{(z', z'') \in U_1 \times U_2 : f(z', z'') = f(z)\right\}$$

coincides with the graph of the of the function g, that is,

$$\left\{(z', z'') \in U_1 \times U_2 : z' = g(z'')\right\}.$$

The proof is analogous to the real case, and can be found in [7, Proposition 1.1.11].

Corollary 2.19 (Inverse function theorem). *Let $U \subset \mathbb{C}^n$ be an open set and let $h: U \to \mathbb{C}^n$ be a holomorphic function. Let $z \in U$ be such that* $\det(J_h(z)) \neq 0$.
Then f is a biholomorphism at z.

Proof. Let us use the Implicit function theorem for $k = n$, and $f(z', z'') = h(z') - z''$, for $z', z'' \in U \subset \mathbb{C}^n$. The assumptions of the theorem are satisfied at the point $(z, h(z)) \in \mathbb{C}^{n+k}$, since the Jacobian of f at $(z, h(z))$ has the following block form

$$J_f(z, h(z)) = \Big(J_h(z) \Big| -Id\Big).$$

Implicit function theorem implies that there are open sets $U_1, U_2 \subset U \subset \mathbb{C}^n$ containing z and a holomorphic function $g : U_2 \to U_1$ such that for $z' \in U_1$, $z'' \in U_2$ we have

$$h(z') = z'' \iff z' = g(z'').$$

□

Remark 2.20. Let $f = (f_1, \ldots, f_n): \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic function. Then $\det J_f(z): \mathbb{C}^n \to \mathbb{C}$ is also a holomorphic function. In particular

$$Z := \{z \in \mathbb{C}^n \mid \det(J_f)^{-1}(0)\}$$

is a closed subset and f is a biholomorphism away from Z.
More generally, the locus where f has rank $\le k$ is a closed subset.

3 Complex Manifolds

3.1 Definition and examples

Definition 3.1. A **complex manifold** (or holomorphic manifold) of dimension n is a Hausdorff topological space X with a countable open cover $\mathcal{U} = \{U_\alpha\}$ and homeomorphisms $\phi_\alpha : U_\alpha \longrightarrow \mathbb{C}^n$ such that the transition functions

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap U_\beta)$$

are biholomorphisms.
The pair (U_α, ϕ_α) is called **complex chart** and the set $\{(U_\alpha, \phi_\alpha)$ is called **complex atlas** or **complex structure**. The n comoponents of the function ϕ_α are called **coordinates** on $U_\alpha \subset X$.

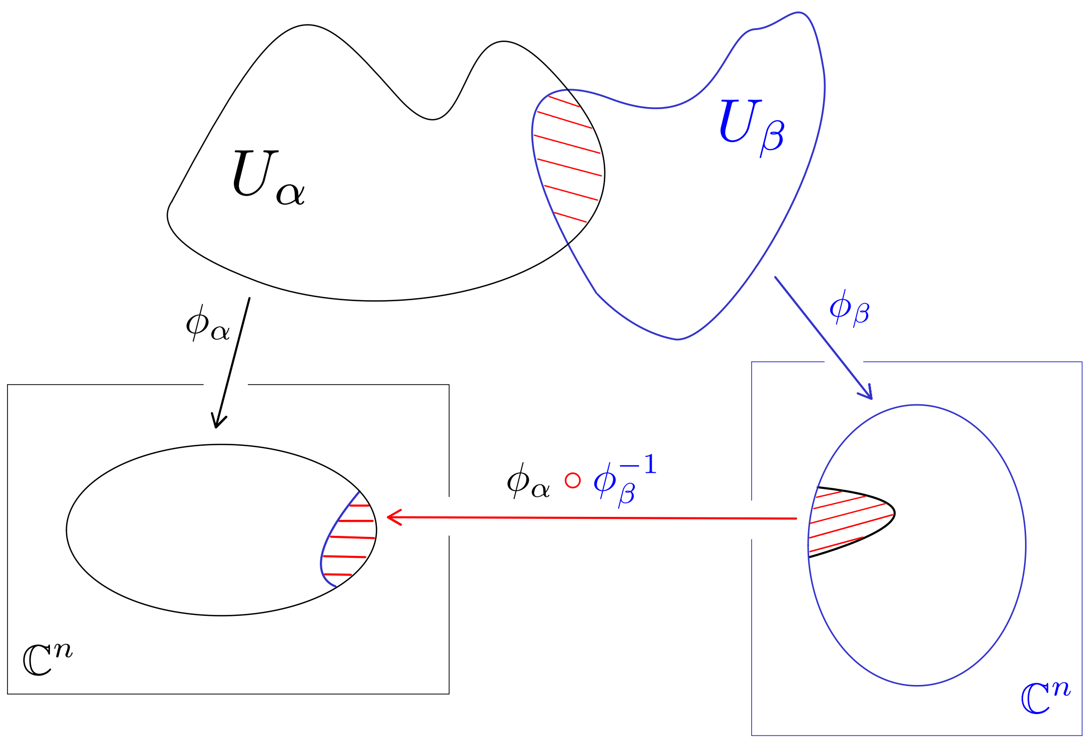

Figure 2: Compatibility of charts in the definition of complex manifold

Note that the real dimension of a complex manifold of dimension n is $2n$.

Example 3.2.

- An open subset $U \subset \mathbb{C}^n$ is a complex manifold or, more generally, an open set of a complex manifold is also a complex manifold.
- If X and Y are complex manifolds then $X \times Y$ is a complex manifold.

Example 3.3 (Projective space). Let $V = \mathbb{C}^{n+1}$ with coordinates $z_0, \ldots, z_n$ and let $V^* = V \setminus \{0\}$. Consider the relation on V^*, given by

$$v \sim w \qquad \text{if} \qquad \exists \lambda \in \mathbb{C}^* \text{ such that } v = \lambda \cdot w.$$

Let $\mathbb{P}^n = V^*/\sim$ with quotient map $\pi\colon V^* \to \mathbb{P}^n$ and endowed with the quotient topology. A point $x \in \mathbb{P}^n$ can be written as an $(n+1)$-tuple $[x_0\colon \dots : x_n]$ so that $x_i \neq 0$ for some i. Two $(n+1)$-tuples $[x_0\colon \dots : x_n]$ and $[y_0\colon \dots : y_n]$ define the same point on $\mathbb{P}^n$ if and only if there is $\lambda \in \mathbb{C}^*$ such that $x_i = \lambda y_i$ for all $i = 0, \dots, n$.
Let $U_i = \{[x_0\colon \dots : x_n] \in \mathbb{P}^n \mid x_i \neq 0\}$. Note that the inequality $x_i \neq 0$ is well-defined, even though the value x_i is ambiguous.
Next, we define $\phi_i : U_i \to \mathbb{C}^n$ via

$$\phi_i([x_0\colon \dots : x_n]) = \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right).$$

Again, the quotients on the right hand side are all well-defined, even though the individual values of x_i are not.
To check compatibility of the charts (U_i, ϕ_i) and (U_j, ϕ_j), let us consider the case $i = 0$ and $j = 1$. Note that

$$\phi_0(U_0 \cap U_1) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 \neq 0\}$$

and

$$\phi_1 \circ \phi_0^{-1}(x_1, x_2, \dots, x_n) = \phi_1\Big([1 : x_1 : x_2 : \dots : x_n]\Big) =$$

$$\phi_1\left(\left[\frac{1}{x_1} : 1 : \frac{x_2}{x_1} : \dots : \frac{x_n}{x_1}\right]\right) = \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right).$$

Thus $\phi_1 \circ \phi_0^{-1}$ is a biholomorphism. Clearly the same result holds for $\phi_i \circ \phi_j^{-1}$ for any i, j.
Thus, $\mathbb{P}^n$ is a complex manifold, called n-dimensional **projective space**. It is sometimes denoted by $\mathbb{CP}^n$ or $\mathbb{P}^n_{\mathbb{C}}$ to stress that the construction starts with a vector space over $\mathbb{C}$.

Proposition 3.4 (Properties of projective space).

1. *The space $\mathbb{CP}^n$ is compact for all n.*
2. *An alternative description of $\mathbb{CP}^n$ is*

$$\mathbb{CP}^n = \{l \subset \mathbb{C}^{n+1} : l \textit{ is a } 1\textit{-dimensional linear subspace}\}.$$

3. *The space $\mathbb{CP}^1$ is homeomorphic to S^1.*

We leave the proof as an exercise.

Example 3.5 (Complex tori). Let $\Lambda = \mathbb{Z}^{2n} \subset \mathbb{C}^n$ be the natural inclusion. Let $X = \mathbb{C}^n/\Lambda$ with quotient map $q\colon \mathbb{C}^n \to X$ endowed with the quotient topology. Note that X is compact. For $x \in X$ let $y \in q^{-1}(x) \subset \mathbb{C}^n$. Then for $D_x := D(y, (1/4, \dots, 1/4))$ we have that the map

$$q_x := q|_{D_x} : D_x \to q(D_x)$$

is a homeomorphism. That is, (D_x, q_x) is a chart of X. By compactness of X it can be covered with finitely many of these, and one checks they define a complex structure.
More generally, let $\Lambda \subset \mathbb{C}^n$ be a lattice of rank $2n$, that is, Λ is a group under addition, abstractly isomorphic to $\mathbb{Z}^{2n}$, and the $2n$ generators of Λ form a real basis of $\mathbb{C}^n = \mathbb{R}^{2n}$. Then $X = \mathbb{C}^n/\Lambda$ is a compact complex manifold, called **complex torus**.
For the case $n = 1$, we obtain a complex curve of genus 1, in the sense that its underlying real manifold has genus 1 (see Figure 3). Such curves are also called **ellliptic curves**.

Definition 3.6. A continuous map $f\colon X \to Y$ between complex manifolds is said to be **holomorphic** if for all $y \in Y$, there is a complex chart (V_y, ψ_y), with $y \in V_y$, such that for all $x \in f^{-1}(y)$, there is a chart (U_x, ϕ_x), with $x \in U_x$, such that $\psi_y \circ f \circ \phi_x^{-1}$ is holomorphic.

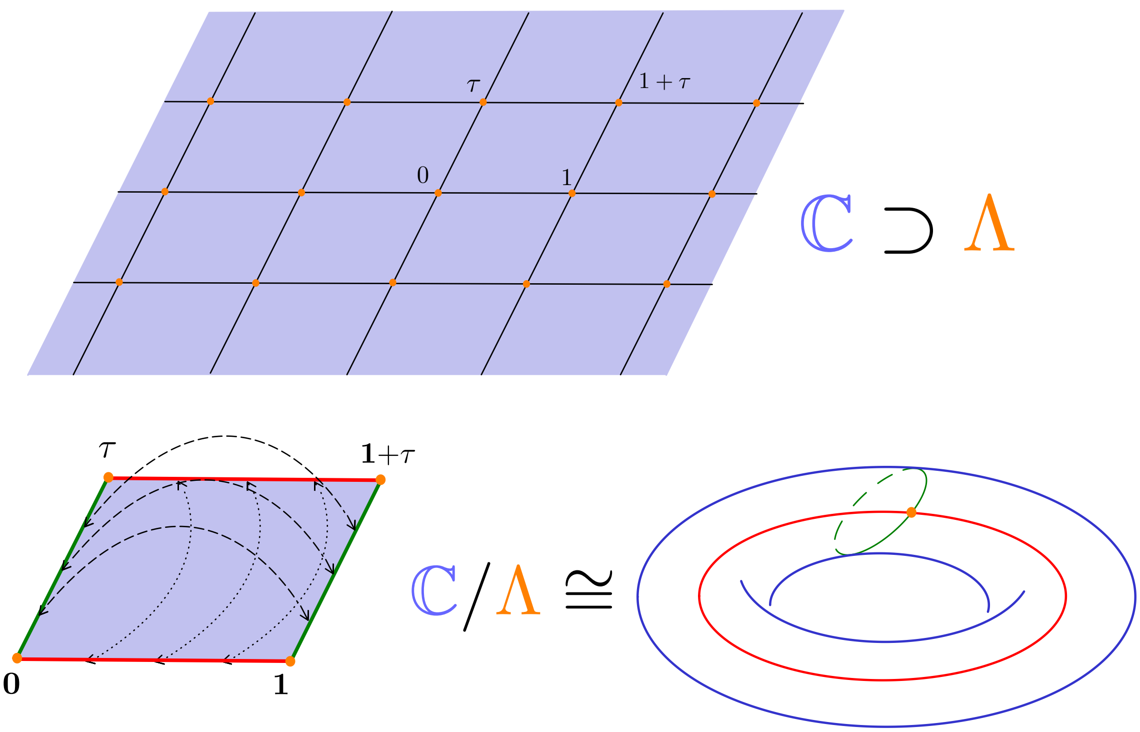

Figure 3: Obtaining an elliptic curve as a quotient of $\mathbb{C}$ by a lattice Λ

It is easy to check that the definition above does not depend on the choice of the charts.
Using the notation above, we define the Jacobian of f at x by taking the Jacobian of $\psi_y \circ f \circ \phi_x^{-1}$.
A holomorphic function on X is just a holomorphic function $f : X \to \mathbb{C}$.

Theorem 3.7. *Let X be a complex manifold and let X be compact and connected. Then any holomorphic function $f : X \longrightarrow \mathbb{C}$ is constant.*

For the proof of this we need:

Proposition 3.8 (Maximum modulus principle). *Let $U \subset \mathbb{C}^n$ be open and $f : U \longrightarrow \mathbb{C}$ be holomorphic. If $|f|$ attains a maximum on U, then f is constant on U.*

Proof of Proposition 3.8. We prove the case $n = 1$, the general case is analog. Assume $|f|$ attains a maximum at $z_0 \in U$. Without loss of generality assume that $z_0 = 0$. Let $\overline{D(0, \epsilon)} = \overline{D} \subset U$. Then

$$\begin{aligned} |f(0)| &= \left| \frac{1}{2\pi} \int_{\partial \overline{D}} \frac{f(z)}{z} \,\mathrm{d}z \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(\epsilon \cdot e^{i\theta})| \,\mathrm{d}\theta \\ &\leq \frac{1}{2\pi} \int_{0^{2\pi}} |f(0)| \,\mathrm{d}\theta \\ &= |f(0)|, \end{aligned}$$

where in the second step we used the monotonicity of the integral and in the third step we used that $|f|$ attains a minimum at z_0. Because the left and right side of this chain of inequalities are the same, all "$\leq$" must really be "=". Therefore, $|f(\epsilon e^{i\theta})| = |f(0)|$ for $\theta \in [0, 2\pi]$. That is: $|f|$ is constant on U.
It remains to show that this implies that f is constant. Write $f = u + iv$ and without loss of generality assume that $|f| = 1$. Then

$$0 = \frac{\partial}{\partial x}(u^2 + v^2) = u_x u + v_x v, \tag{$*$}$$

$$0 = \frac{\partial}{\partial y}(u^2 + v^2) = u_y u + v_y v = -v_x u + u_x v, \tag{$**$}$$

where in the last step we used the Cauchy-Riemann equations for f. Thus

$$u_x = u_x(u^2 + v^2) = u_x uu + u_x vv = u_x uu + v_x vu = u\underbrace{(uu_x + v_x v)}_{=0} = 0,$$

where in the third step we used $(**)$ and in the last step we used $(*)$. Analogously we find that $v_x = 0$, so f is constant. $\square$

3.2 Almost complex structures and the tangent bundle

Definition 3.9 (Tangent space). Let $K \in \{\mathbb{R}, \mathbb{C}\}$ and M be a K-manifold. Then

$$T_{\mathbb{R},x}M := \{[\gamma] \text{ s.t. } \gamma : (-\epsilon, \epsilon) \to M \text{ curve for some } \epsilon > 0 \text{ s.t. } \gamma(0) = x\},$$

where $[\cdot]$ denotes the equivalence class with respect to the equivalence relation $\sim$ defined by

$$\begin{aligned} \gamma \sim \delta :\Leftrightarrow\ &\text{ex. } (-\widetilde{\epsilon}, \widetilde{\epsilon}) \text{ on which } \gamma \text{ and } \delta \text{ are both defined} \\ &\text{and } \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\phi\gamma(t) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\phi\delta(t) \text{ for a chart } (U, \phi) \text{ around } x \end{aligned}$$

The vector space structure is given by

$$\lambda \cdot [\gamma] + \mu \cdot [\delta] := \left[\phi^{-1}(\lambda \cdot \phi\gamma + \mu \cdot \phi\delta)\right]$$

for $\lambda, \mu \in K$ and $[\gamma], [\delta] \in T_{\mathbb{R},x}M$.

Note that this definition defined a K-vector space structure, i.e. if the manifold is complex, then we can multiply tangent vectors by complex numbers.
If $K = \mathbb{C}$, i.e. M is complex, and $(U, \phi = (z_1, \ldots, z_n))$ is a chart around $x \in M$ write $z_i = x_i + iy_i$. Then

$$T_{\mathbb{R},x}M = \operatorname{span}_{\mathbb{R}}\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right\}.$$

Definition 3.10. The map

$$\begin{aligned} J_x : T_{\mathbb{R},x}M &\to T_{\mathbb{R},x}M \\ [\gamma] &\mapsto \left[\phi^{-1}(i \cdot \phi\gamma)\right] \end{aligned}$$

satisfying $J_x^2 = -\mathrm{Id}$ is called *almost complex structure induced by the complex structure.*

Definition 3.11 (Complexified tangent space). The complex vector space

$$T_{\mathbb{C},x}M := T_{\mathbb{R},x}M \otimes_{\mathbb{R}} \mathbb{C}$$

is called complexified tangent space. Defining

$$\frac{\partial}{\partial z_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i \cdot \frac{\partial}{\partial y_i}\right), \text{ and } \frac{\partial}{\partial \overline{z_i}} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i \cdot \frac{\partial}{\partial y_i}\right),$$

we have that

$$\begin{aligned} T_{\mathbb{C},x}M &= \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right\} \\ &= \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_i}}\right\} \end{aligned}$$

and we define

$$\begin{aligned} T_x^{1,0}M &:= \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_i}\right\}, \\ T_x^{0,1}M &:= \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial \overline{z_i}}\right\}. \end{aligned}$$

The space $T_{\mathbb{C},x}M$ is called the *holomorphic tangent space* and $T_{\mathbb{C},x}M$ is called the *anti-holomorphic tangent space*.

Remark 3.12. Let M be a complex manifold.

1. The map

$$\begin{aligned} F : T_{\mathbb{R},x}M &\longrightarrow T_x^{1,0}M \\ \frac{\partial}{\partial x_i} &\longmapsto \frac{\partial}{\partial z_i} \end{aligned}$$

   is an isomorphism of $\mathbb{C}$-vector spaces. In particular, if n is the complex dimension of M, then the complex dimension of $T_{\mathbb{C},x}M$ is $2n$, and the complex dimension of $T_x^{1,0}M$ is n.

2. If one extends $J_x : T_{\mathbb{R},x}M \longrightarrow T_{\mathbb{R},x}M$ complex linearly to a map $J_x : T_{\mathbb{C},x}M \longrightarrow T_{\mathbb{C},x}M$, then

$$\begin{aligned} T_x^{1,0}M &= \text{the } i\text{-eigenspace of } J_x, \\ T_x^{0,1}M &= \text{the } (-i)\text{-eigenspace of } J_x. \end{aligned}$$

Definition 3.13. Let $f : X \longrightarrow Y$ be holomorphic. The *Jacobian* J_f of f at $x \in X$ is the complex matrix representing the linear map $\mathrm{d}f_{\mathbb{C}} : T_x^{1,0}X \longrightarrow T_{f(x)}^{1,0}Y$ in the bases $\frac{\partial}{\partial z_i}$ on $T_x^{1,0}X$ and $T_{f(x)}^{1,0}Y$.

Definition 3.14. A holomorphic map $f : X \longrightarrow Y$ is a **submersion** (resp. **immersion**) if $\dim X \geq \dim Y =: r$ (resp. $r := \dim X \leq \dim Y$) at every point $x \in X$ the Jacobian J_f of f has maximal rank r.
An immersion is an **embedding** if $f : X \longrightarrow f(X)$ is a homeomorphism.

Example 3.15. Let $\mathbb{Z}^4 \subset \mathbb{C}^2$ be the standard lattice and let $T^4 = \mathbb{C}^2/\mathbb{Z}^4$. Denote by $q : \mathbb{C}^2 \longrightarrow T^4$ the quotient map. As in Example 3.5, T^4 is a complex manifold. Let $\lambda \in \mathbb{C}$ and consider the immersion $f : \mathbb{C} \longrightarrow \mathbb{C}^2$ given by

$$x \longmapsto (x, \lambda x).$$

The composition $q \circ f : \mathbb{C} \longrightarrow T^4$ is also an immersion, as

$$J_{q \circ f} = J_q \cdot J_f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot (1, \lambda),$$

where we used the chain rule for the differential in the first step, and we computed Jacobians with respect to the obvious charts on $\mathbb{C}$ and $\mathbb{C}^2$, inducing an obvious chart on T^4.

If $f\colon X \to Y$ is an embedding such that $f(X)$ is closed in Y then we say that X is a **closed submanifold** of Y. The **codimension** of X is $\dim Y - \dim X$.
The following result provides a way to check if a closed subset of a complex manifold is a submanifold.

Theorem 3.16. *Let Y be a manifold of dimension n and $i\colon X \to Y$ be a closed submanifold of codimension k. Then for all $p \in i(X)$ there is a biholomorphism $\phi : U \to \phi(U) \subset \mathbb{C}^n$ defined in a small open neighborhood U of p such that*

$$\phi\big(i(X) \cap U\big) = \{(z_1, ..., z_n) \in \phi(U) : z_1 = ... = z_k = 0\}.$$

Conversely, if $X \subset Y$ is a closed subset such that for all $x \in X$ there is an open subset $x \in U \subset Y$ and a submersion $f\colon U \to f(U) \subset \mathbb{C}^k$ such that $X \cap U = f^{-1}(0)$, then X is a closed submanifold of codimension k.

The proof uses the Implicit Function theorem Theorem 2.18, and is analogous to the real case.

Example 3.17 (Complete intersections in $\mathbb{P}^n$). Let $X = \mathbb{P}^n$ and let F_i be homogeneous polynomials of degree d_i, $i = 1, ..., k$, $k \leq n$. Let the Jacobian

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{1\leq i\leq k\\ 0\leq j\leq n}} \tag{$*$}$$

have the maximal rank k at each point $x \in \mathbb{C}^{n+1} \setminus \{0\}$ such that $F_1(x) = ... = F_k(x) = 0$.
Consider the following closed subset of $\mathbb{P}^n$:

$$V = \{x \in \mathbb{P}^n : F_1(x) = ... = F_k(x) = 0\}.$$

We claim that V is a submanifold of $\mathbb{P}^n$ of codimension k. Such a manifold is called a complete intersection in $\mathbb{P}^n$.
It is enough to verify the claim in each open set $U_\ell \subset \mathbb{P}^n$, where $\{U_\ell\}_{0\leq \ell\leq n}$ is an open cover of X defined in Example 3.3. Using the chart map $\phi_\ell : U_\ell \to \mathbb{C}^n$, we identify $V \cap U_\ell$ with the set of points $(x_0, ..., x_{\ell-1}, x_{\ell+1}, ..., x_n) \in \mathbb{C}^n$ satisfying

$$F_i(x_0, ..., x_{\ell-1}, 1, x_{\ell+1}, ..., x_n) = 0, \quad i = 1, ..., k.$$

Now, if the submatrix of the Jacobian of F given by

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,...,k\\ j=0,...,\ell-1,\ell+1,....n}} \tag{$**$}$$

has rank k at a point $x = (x_0, ..., x_{\ell-1}, 1, x_{\ell+1}, ..., x_n) \in \mathbb{C}^{n+1}$, then the functions $G_i = F_i|_{x_\ell=1}$, $i = 1, ..., k$, form a submersion $\mathbb{C}^n \to \mathbb{C}^k$ near x. In this case, Theorem 3.16 implies that $V \cap U_\ell$ is a submanifold of codimension k locally near $[x_0 : \ldots : x_{\ell-1} : 1 : x_{\ell+1} : \ldots : x_n] \in U_i$, and we are done.
It remains to prove that the matrix $(**)$ has rank k at each x such that $[x_0 : \ldots : x_{\ell-1} : 1 : x_{\ell+1} : \ldots : x_n] \in V \cap U_i$. Using the fact that F_i is homogenous of degree d_i, we obtain

$$\sum_{j=0}^{n} x_j \frac{\partial F_i}{\partial x_j}(x) = d_i F_i(x) = 0 \implies$$

$$\frac{\partial F_i}{\partial x_\ell}(x) = -\sum_{\substack{j=0\\ j\neq \ell}}^{n} x_j \frac{\partial F_i}{\partial x_j}(x).$$

This means that the ℓ-th column of the Jacobian matrix $(*)$ is a linear combination of the other columns. Therefore, by removing the ℓ-th column from the Jacobian, we cannot lower the rank of the matrix.

4 Holomorphic vector bundles

4.1 Definition and examples

Before defining *holomorphic* vector bundles, we first remember the definition of *vector bundle*:

Definition 4.1. Let K be a field and E, X be Hausdorff spaces. Let $\pi : E \to X$ be continuous. Then E is called a *K-vector bundle of rank r* if:

1. for all $p \in X$ we have that $E_p := \pi^{-1}(p)$ is a K-vector space of dimension r. The space E_p is called the *fibre* over p.

2. for all $p \in X$ there exists a neighbourhood U of p and a homeomorphism $h : \pi^{-1}(U) \to U \times K^r$ such that $h(E_p) \subset \{p\} \times^r$ and h^p is defined as

$$h^p : E_p \xrightarrow{h} \{p\} \times K^r \xrightarrow{p_2} K^r$$

   is a K-vector space isomorphism, where p_2 denotes the projection onto the second component. The pair (U, h) is called a local trivialisation of E.

The space E is called the *total space* of the vector bundle, X is called the *base space*. If $r = 1$, then E is called a *line bundle*.

Roughly speaking, a smooth (respectively holomorphic) vector bundle is then a bundle in which all the maps appearing in the above definition are smooth (respectively holomorphic). To be precise:

Definition 4.2. A K-vector bundle $\pi : E \to X$ is said to be smooth (respectively holomorphic) if X and E are smooth (respectively complex) manifolds and π and h from the previous definition are smooth (respectively holomorphic).

Remark 4.3. Let $(U_\alpha, h_\alpha), (U_\beta, h_\beta)$ be local trivialisations of a K-vector bundle. The induced map

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\to \mathrm{GL}(r, K) \\ p &\mapsto H_\alpha^p \circ (h_\beta^p)^{-1} \end{aligned}$$

is called *transition function*. It satisfies the compatibility conditions:

1. $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = \mathrm{Id}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ and

2. $g_{\alpha\alpha} = \mathrm{Id}$ on U_α.

If the local trivialisations of the K-vector bundle are smooth (holomorphic), then the transition functions are smooth (holomorphic). The converse of this statement is not literally true, but the next proposition serves as some sort of a converse to this statement.

Proposition 4.4. *Given a covering $\{U_\alpha\}$ of X and for each α, β a smooth (respectively holomorphic) function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, K)$ satisfying conditions 1. and 2. from Remark 4.3. Then there exists a smooth (respectively holomorphic) vector bundle with these transition functions.*

Proof. We give the proof idea. Let

$$\widetilde{E} := \bigcup_\alpha U_\alpha \times K^r,$$

where $\cup$ denotes the disjoint union. On $\widetilde{E}$ we define the equivalence relation $\sim$ as follows: for $(x, v) \in U_\beta \times K^r$ and $(y, w) \in U_\alpha \times K^r$ let

$$(x, v) \sim (y, w) \quad :\Leftrightarrow \quad y = x \text{ and } w = g_{\alpha\beta}(x)v.$$

Set $E := \widetilde{E}/\sim$ and

$$\begin{aligned} \pi : E &\longrightarrow X \\ [(x, v)] &\mapsto x. \end{aligned}$$

One checks that E is well-defined and has the trivialisations with transition functions as claimed in the proposition. □

Definition 4.5. Let X be a complex manifold and let $\pi\colon E \longrightarrow X$ and $\pi'\colon F \longrightarrow X$ be holomorphic vector bundles on X. A **morphism of vector bundles** $\phi\colon E \longrightarrow F$ over X is a holomorphic morphism such that $\pi = \pi' \circ \phi$ and such that, for each $x \in X$, the induced map $\phi(x)\colon E(x) \longrightarrow F(x)$ is linear and the rank of $\phi(x)$ is independent of $x \in X$.

Example 4.6 (Trivial bundle). Let X be a complex manifold and let $E = X \times \mathbb{C}^r$. Then E is a holomorphic vector bundle of rank r. E is called **trivial bundle** of rank r.
For $r = 0$, we obtain the zero vector bundle, i.e. the vector bundle $E = X$ whose fibers are all zero dimensional.
For $r = 1$, we get the trivial line bundle over X, which is denoted by $\mathcal{O}_X$ or $\underline{\mathbb{C}}$.

Example 4.7 (Algebra of vector bundles). Let E and F be vector bundles over a complex manifold X of rank r and s respectively. Then for every "nice" operation on the vector spaces, there is a corresponding operation on the vector bundles. Here are a few examples:

- The **direct sum** $E \oplus F$ is the holomorphic vector bundle over X of rank $r + s$ whose fiber over $x \in X$ is $E(x) \oplus F(x)$.
- the **tensor product** $E \otimes F$ is the holomorphic vector bundle of rank rs whose fiber over $x \in X$ is $E(x) \otimes F(x)$.
- the p-th **exterior power** $\Lambda^i E$ of E is the holomorphic vector bundle whose fiber over $x \in X$ is the exterior power $\Lambda^i E(x)$. In particular, the **determinant bundle** of E is $\det E := \Lambda^r E$. Note that $\det E$ is a line bundle.
- The dual bundle E^* of E is the holomorphic vector bundle whose fiber over $x \in X$ is the dual $E(x)^*$.
- If $\phi\colon E \longrightarrow F$ is a morphism of vector bundles, then the **kernel** $\ker \phi$ of ϕ is the holomorphic vector bundle whose fiber over $x \in X$ is the kernel of $\phi(x)\colon E(x) \longrightarrow F(x)$. The **cokernel** $\operatorname{coker}\phi$ of ϕ is the holomorphic vector bundle whose fiber over $x \in X$ is the kernel of $\phi(x)\colon E(x) \longrightarrow F(x)$.

Proposition 4.8. *The set*

$$\mathcal{O}(-1) := \{(x, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \in x\}$$

has a canonical structure of a holomorphic vector bundle over $\mathbb{CP}^n$.

Proof. Let $\pi : \mathcal{O}(-1) \longrightarrow \mathbb{CP}^n$ be the projection onto the first component. Let $\mathbb{CP}^n = \cup_{i=0}^n U_i$ be the standard affine cover of $\mathbb{CP}^n$. Define

$$\begin{aligned} \psi_i : \pi^{-1}(U_i) &\longrightarrow U_i \times \mathbb{C} \\ (x, v) &\mapsto (x, v_i), \text{ where } v_i \text{ denotes the } i\text{-th entry of } v. \end{aligned}$$

Exercise 4.9. *Check that the transition maps are given by*

$$\begin{aligned} \psi_{ij}([x_0 : \cdots : x_n]) : \mathbb{C} &\longrightarrow \mathbb{C} \\ v &\mapsto \frac{x_i}{x_j} v \end{aligned}$$

for $[x_0 : \cdots : x_n] \in U_i \cap U_j$.

Thus, the ψ_i define a complex structure on $\mathcal{O}(-1)$, and it follows from the definition that π and ψ_i are holomorphic with respect to this structure. Thus, $\mathcal{O}(-1)$ is a holomorphic vector bundle. $\square$

Definition 4.10. The line bundle $\mathcal{O}(-1)$ is called the **tautological line bundle** on $\mathbb{P}^n$. The line bundle $\mathcal{O}(1)$ is defined as the dual $\mathcal{O}(-1)^*$ of $\mathcal{O}(-1)$. Furthermore, for any $k > 0$, we define

$$\mathcal{O}(k) := \mathcal{O}(1)^{\otimes k} = \mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1),$$

$$\mathcal{O}(-k) := \mathcal{O}(-1)^{\otimes k} = \mathcal{O}(-1) \otimes \cdots \otimes \mathcal{O}(-1)$$

We also define $\mathcal{O}(0)$ to be the trivial line bundle $\mathcal{O}_{\mathbb{P}^n}$. Sometimes, we include the subscript $\mathbb{P}^n$ in the notation $\mathcal{O}_{\mathbb{P}^n}(k)$ to indicate which projective space the line bundle lives over.

Exercise 4.11. *Find the transition functions for the bundle $\mathcal{O}(1)$ in a trivialisation of your choice.*

The bundles $\mathcal{O}(k)$ are very important as building blocks of other holomorphic vector bundles. On $\mathbb{CP}^1$ the situation is particularly simple, and every holomorphic vector bundle is just a sum of these line bundles:

Theorem 4.12 (Grothendieck lemma). *Every holomorphic vector bundle E on $\mathbb{CP}^1$ is isomorphic to a holomorphic vector bundle of the form $\bigoplus \mathcal{O}(a_i)$.*

Proof. [7, Corollary 5.2.8]. $\square$

In general, the situation is much more complicated. For example, there are many open questions about holomorphic vector bundles on $\mathbb{CP}^2$.

Definition 4.13. Let $f : Y \to X$ be holomorphic and $\pi : E \to X$ be a rank r holomorphic vector bundle. The set

$$f^*E := \{(y, v) \in Y \times E : f(y) = \pi(e)\}$$

is called the *pullback of E to Y*.

Proposition 4.14. *The set f^*E has a canonical structure of a rank r vector bundle over Y.*

Proof. Let $\{U_i\}$ be a cover of X and $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ be a trivialisation of E. Define $\pi' : f^*E \to Y$ as $\pi'(y, v) = y$. Then $\{f^{-1}(U_i)\}$ is a cover of Y and trivialisations of f^*E are given by

$$\begin{aligned} \psi'_i : (\pi')^{-1}(f^{-1}(U_i)) &\to f^{-1}(U_i) \times \mathbb{C}^r \\ (y, v) &\mapsto (y, p_2(\psi_i(v)), \end{aligned}$$

where $p_2 : U_i \times \mathbb{C}^r \to \mathbb{C}^r$ denotes the projection onto the second component. As in the proof of Proposition 4.8 that the maps ψ'_i define a complex structure on f^*E making it a holomorphic vector bundle. $\square$

Definition 4.15. Let Y be a complex manifold and let $X \subset Y$ be a complex submanifold, with inclusion map $i\colon X \to Y$. Then $E|_X := i^*E$ is called the *restriction of E to X*.

Definition 4.16. Let $\pi\colon E \to X$ be a holomorphic vector bundle over a complex manifold X. A **section** of E is a holomorphic morphism $s\colon X \to E$ such that $\pi \circ s = \mathrm{Id}_X$. The space of holomorphic sections of E is denoted by $H^0(X, E)$.

Remark 4.17. Every holomorphic vector bundle $E \to X$ admits the **zero section**, which in a trivialization $E|_{U_i} \cong U_i \times \mathbb{C}^r$ is given by $x \mapsto (x, 0)$.

Exercise 4.18. *A holomorphic line bundle $L \to X$ is isomorphic to the trivial line bundle if and only if it admits a section that is nowhere zero.*

Definition 4.19. Let $E \to X$ be a rank r holomorphic vector bundle. Sections $s_1, \ldots, s_r \in H^0(X, E)$ with the property that $s_1(x), \ldots, s_r(x)$ are a basis of E_x for all $x \in X$ are called a *frame* of E. If $U \subset X$ open, then a frame of $E|_U$ is called a *local frame*.

Local frames exist for all holomorphic vector bundles, frames need not exist.

Example 4.20 (Tangent bundle). Let X be a complex manifold of dimension n and for each $x \in X$, let $T_x^{1,0}X$ be the holomorphic tangent space of M at x. Let $T_X^{1,0} := \cup_{x\in X} T_x^{1,0}X$ with the morphism $\pi\colon T_X^{1,0} \to X$ such that $\pi^{-1}(x) = T_x^{1,0}X$ for each $x \in X$. We want to show that $T_X^{1,0}$ is a vector bundle of dimension n. Let $\{(U_\alpha, \phi_\alpha)\}$ be a holomorphic atlas. Then, $\phi_\alpha(U_\alpha) \subset \mathbb{C}^n$ and for each $x \in U_\alpha$, the Jacobian of ϕ_α at x defines a linear map

$$T_x^{1,0}X \to T_{\phi_\alpha(x)}^{1,0}\phi_\alpha(U_\alpha) = \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right\}$$

which induces a map

$$\psi_\alpha\colon \pi^{-1}(U_\alpha) = \bigcup_{x\in U_\alpha} T_x^{1,0}X \to U_\alpha \times \mathbb{C}^n.$$

One checks (as in the proof of Proposition 4.8 or Proposition 4.14) that this defines a complex structure on $T_X^{1,0}$. Thus, $T_X^{1,0}$ is a holomorphic vector bundle, called the **tangent bundle** of X. Note that $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$ define a local frame.

Example 4.21. The **cotangent bundle** Ω_X^1 of X is the dual of T_X. For each $p \geq 1$, we denote $\Omega_X^p := \Lambda^p \Omega_X^1$. For the case $p = n$, the vector bundle Ω_X^n is of special importance and is called the **canonical line bundle** of X. A section of $T_X^{1,0}$ is called a **holomorphic vector field** on X. A section of Ω_X^p is called a **holomorphic** p**-form** on X.

4.2 Almost complex structures and the complexified tangent bundle revisited

Definition 4.22. Let X be a differentiable manifold. An **almost complex structure** on X is a differentiable vector bundle isomorphism $J\colon T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ such that $J^2 = -\mathrm{Id}$. The pair (X, J) is called an *almost complex manifold*.

Remark 4.23. Here, *smooth* means that J represented in a local trivialisation (U_i, ψ_i) of the real vector bundle TX is a smooth map $U_i \to \mathrm{GL}(n, \mathbb{R})$. Equivalently, J is a smooth section of the $\mathbb{R}$-vector bundle $\mathrm{End}(TX) \cong TX \otimes_{\mathbb{R}} (T^*X)$.

Proposition 4.24. *A complex manifold X admits a natural almost complex structure.*

Proof. Let $x \in X$ and (U, ϕ) be a complex chart around x The map

$$\begin{aligned} J_x : T_{\mathbb{R},x}X &\to T_{\mathbb{R},x}X \\ [\gamma] &\mapsto [\phi^{-1}(i \cdot \phi\gamma)] \end{aligned}$$

from Definition 3.10 satisfies $J_x^2 = -\mathrm{Id}$. It remains to check that J is smooth in x. Writing $\phi = (z_1, \dots, z_n)$ and $z_k = x_k + iy_k$ we have that $T_{\mathbb{R},y}X = \mathrm{span}_{\mathbb{R}}\left\{\frac{\partial}{\partial x_k}(y), \frac{\partial}{\partial y_k}(y)\right\}$ and

$$J_y = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}$$

in this basis. This is constant in y, in particular depends smoothly on y. ☐

Proposition 4.25. *If a smooth manifold admits an almost complex structure, then it is orientable.*

Proof. We only give the proof for *complex manifolds* and leave it as an exercise to extend it to the case of almost complex manifolds.
One definition of orientation is an atlas in which the Jacobians of the the transition functions have positive determinant. A complex atlas satisfies this condition, as we will now check. Let (U_α, ϕ_α) and (U_β, ϕ_β) be complex charts. Then $\mathrm{d}\,(\phi_\alpha\phi_\beta^{-1}) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is complex linear by exercise sheet 1. Let $L : \mathbb{C}^n \to \mathbb{C}^n$ denote the corresponding complex linear map. Then

$$\det{}_{\mathbb{R}}(\mathrm{d}\,(\phi_\alpha\phi_\beta^{-1})) = |\det{}_{\mathbb{C}}L|^2 > 0,$$

where in the second step we used exercise sheet 2. This proves the claim. ☐

Definition 4.26. Let (X, J) be an almost complex manifold. Extend J complex linearly to $J : T_{\mathbb{C}}X \to T_{\mathbb{C}}X$ and let

$$\begin{aligned} T^{0,1} &:= i\text{-eigenspace of } J \subset T_{\mathbb{C}}X, \\ T^{1,0} &:= (-i)\text{-eigenspace of } J \subset T_{\mathbb{C}}X. \end{aligned}$$

Note that on complex manifolds we made a different definition for the holomorphic and anti-holomorphic tangent bundle, and we saw in Remark 3.12 that their description as eigenspaces of J was a *consequence* of this definition. Because of this, we mimic this on almost complex manifolds in Definition 4.26, where no other definition of holomorphic tangent bundle exists.

Remark 4.27. By Definition 4.26 we have

$$T^{1,0}X = \operatorname{Ker}(J - i \cdot \mathrm{Id}),$$
$$T^{0,1}X = \operatorname{Ker}(J + i \cdot \mathrm{Id}),$$

so by Example 4.7 they are complex vector bundles. They are not holomorphic vector bundles, because that definition only makes sense on complex manifolds and (X, J) is only assumed to be an almost complex manifold.

Definition 4.28. Let V be a real vector space. The map

$$\overline{(\cdot)} : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$$
$$v \otimes \lambda \mapsto v \otimes \overline{\lambda}$$

is called *conjugation*. Let (X, J) be an almost complex manifold. Then this extends to $\overline{(\cdot)} : T_{\mathbb{C}}X \longrightarrow T_{\mathbb{C}}X$ satisfying $T^{0,1}X = \overline{T^{1,0}X}$.
If $(x_1, \ldots, x_n)$ are local coordinates on X, then let

$$\begin{aligned} \frac{\partial}{\partial z_i} &:= \frac{1}{2}\left(\frac{\partial}{\partial x_i} - iJ\frac{\partial}{\partial x_i}\right), \\ \frac{\partial}{\partial \overline{z_i}} &:= \frac{1}{2}\left(\frac{\partial}{\partial x_i} + iJ\frac{\partial}{\partial x_i}\right), \end{aligned} \tag{$*$}$$

and one checks that

$$T^{1,0}X = \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_i}\right\} = \operatorname{span}_{\mathbb{R}}\left\{\frac{\partial}{\partial z_i}\right\},$$
$$T^{0,1}X = \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial \overline{z_i}}\right\} = \operatorname{span}_{\mathbb{R}}\left\{\frac{\partial}{\partial \overline{z_i}}\right\}.$$

The bundles $T^{1,0}X$ and $T^{0,1}X$ have rank $(\dim X)/2$ viewed as complex vector bundles. In equation $(*)$ above, the vector fields $\frac{\partial}{\partial \overline{z_i}}$ and $\frac{\partial}{\partial z_i}$ respectively form a basis over $\mathbb{R}$, but they are linearly dependent over $\mathbb{C}$.

Here comes a reminder about the Lie bracket (also known as the vector field commutator): Let M be a smooth manifold with local coordinates $x_1, \ldots, x_n$. Let $[\cdot, \cdot] : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ for

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^{n} Y_i \frac{\partial}{\partial y_i}$$

be defined as

$$[X, Y] := \sum_{i=1}^{n} (X(Y_i) - Y(X_i)) \frac{\partial}{\partial x_i}.$$

Then $[X, Y]$ is called the *Lie bracket* of X and Y.

Proposition 4.29. *The Lie bracket has the following properties: for $X, Y, Z \in \mathfrak{X}(M)$, $\lambda, \mu \in \mathbb{R}$, $f \in C^\infty(M)$:*

1. *(Skew-symmetry)* $[X, Y] = -[Y, X]$,
2. *(Bilinearity)* $[\lambda X + \mu Y, Z] = \lambda[X, Z] + \mu[Y, Z]$,
3. *(Jacobi identity)* $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$,
4. $[X, Y]f = X(Y(f)) - Y(X(f))$,

5. *on* $\mathbb{R}^n$: $[X, Y] = X(Y) - Y(X)$,

6. $[fX, gY] = fg[X, Y] + fX(g)Y + gY(f)X$,

7. *if* $x_1, \ldots, x_n$ *are local coordinates, then* $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ *for all* $i, j \in \{1, \ldots, n\}$.

Definition 4.30. Let X be a differentiable manifold. An almost complex structure $J : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ is called **integrable** if the Lie bracket of any two sections of $T_X^{0,1}$ is again a section of $T_X^{0,1}$.

Here the Lie bracket on sections of $T_X^{0,1}$ is induced by the usual Lie bracket of sections $T_{X,\mathbb{R}}$, i.e. the Lie bracket of vector fields on X, via complexification.

Proposition 4.31. *Let X be a complex manifold. Then the almost complex structure $J : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ from Proposition 4.24 is integrable.*

Proof. Each section of $T_X^{0,1}$ has a local expression $\sum_{j=1}^n f_j \frac{\partial}{\partial \bar{z}_j}$, where f_j are C^∞-functions. For any two sections $\sum_{j=1}^n f_j \frac{\partial}{\partial \bar{z}_j}$ and $\sum_{j=1}^n g_j \frac{\partial}{\partial \bar{z}_j}$ of $T_X^{0,1}$, we have

$$\begin{aligned}\left[\sum_{j=1}^n f_j \frac{\partial}{\partial \bar{z}_j}, \sum_{j=1}^n g_j \frac{\partial}{\partial \bar{z}_j}\right] &= \sum_{j=1}^n \sum_{k=1}^n \left[f_j \frac{\partial}{\partial \bar{z}_j}, g_k \frac{\partial}{\partial \bar{z}_k}\right] \\ &= \sum_{j=1}^n \sum_{k=1}^n \left(f_j \frac{\partial g_k}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{z}_k} - g_k \frac{\partial f_j}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{z}_j}\right),\end{aligned}$$

where in the second step we used properties 6 and 7 from Proposition 4.29. The result is evidently a section of $T_X^{0,1}$ as well. □

Remark 4.32. This proof fails for almost complex manifolds. Even though we can still define $\frac{\partial}{\partial \bar{z}_j}$, we can only apply property 7 from Proposition 4.29 if they come from complex coordinates.

Remark 4.33. On an almost complex manifold (X, J) define the *Nijenhuis tensor* as

$$N_J(V, W) := [V, W] + J([JV, W] + [V, JW]) - [JV, JW]$$

for $V, W \in C^\infty(X, TX)$. Then J is integrable if and only if $N_J(V, W) = 0$ for all $V, W \in C^\infty(X, TX)$.

The previous remark is an exercise that is not very hard to prove. The following theorem however is a deep theorem requiring PDE techniques to prove it, and we will not give its proof in this course:

Theorem 4.34. *(Newlander-Nirenberg) Let X be a differentiable manifold. Let $J : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ be an integrable almost complex structure.*
Then X admits a structure of complex manifold that induces J.

4.3 Dolbeault cohomology of a complex manifold

Let X be a complex manifold. By taking the dual, the decomposition $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$ induces a decomposition

$$\Omega^1_{X,\mathbb{C}} := T^*_{X,\mathbb{C}} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

and by taking the k-th exterior power of $\Omega_{X,\mathbb{C}}$, we have

$$\Omega^k_{X,\mathbb{C}} := \bigwedge^k \Omega^1_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega^{p,q}_X$$

where

$$\Omega^{p,q}_X := \bigwedge^p_X \Omega^{1,0}_X \otimes \bigwedge^q \Omega^{0,1}_X$$

is the vector bundle of (p,q)-forms.

Definition 4.35. Let X be a complex manifold. A (p,q)-**form** (or a **form of type** (p,q)) on X is a section of the complex vector bundle $\Omega^{p,q}_X$.

Note that in local coordinates a (p,q)-form on X can be written as

$$\omega = \sum_{i_1,\dots,i_p,j_1,\dots,j_q} \alpha_{i_1,\dots,i_p,j_1,\dots,j_q} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q},$$

where $\alpha_{i_1,\dots,i_p,j_1,\dots,j_q}$ are smooth functions. To simplify the notation, we will denote

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p} \quad \text{and} \quad d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q},$$

so that

$$\omega = \sum_{|I|=p,|J|=q} \alpha_{I,J} dz_I \wedge d\overline{z}_J.$$

Recall that for a real manfiold X, we can consider the exterior differential

$$d\colon C^\infty\left(X, \Omega^k_X\right) \to C^\infty\left(X, \Omega^{k+1}_X\right),$$

where $C^\infty\left(X, \Omega^k_X\right)$ stands for the space of C^∞-sections of the vector bundle Ω^k_X. Recall that the differential satisfies the Leibnitz rule and the property $d \circ d = 0$, i.e. $d(d(\omega)) = 0$ for every $\omega \in C^\infty\left(X, \Omega^k_X\right)$.

Definition 4.36. Coming back to the case when X is a complex manifold, the expression $d \otimes \mathrm{Id}_{\mathbb{C}}$ defines an exterior differential on the complexified cotangent bundle $\Omega^1_{X,\mathbb{C}}$, which we still denote by d:

$$d : C^\infty\left(X, \Omega^k_{X,\mathbb{C}}\right) \to C^\infty\left(X, \Omega^{k+1}_{X,\mathbb{C}}\right).$$

Elements of $C^\infty\left(X, \Omega^k_{X,\mathbb{C}}\right)$ are called complexified k-form on X. Clearly, d still satisfies the Leibnitz rule and the property $d \circ d = 0$. Let $\omega \in C^\infty(X, \Omega^{p,q}_X)$. Then $d\omega \in C^\infty(X, \Omega^{p+q+1}_{X,\mathbb{C}})$. More precisely, if we write

$$\omega = \sum_{|I|=p,|J|=q} \alpha_{I,J} dz_I \wedge d\overline{z}_J,$$

where $\alpha_{I,J}$ are smooth functions, then

$$d\omega = \sum_{|I|=p,|J|=q} \partial(\alpha_{I,J}) \wedge dz_I \wedge d\overline{z}_J + \sum_{|I|=p,|J|=q} \overline{\partial}(\alpha_{I,J}) \wedge dz_I \wedge d\overline{z}_J$$

where, for every smooth function α, we write

$$\partial\alpha := \sum_{j=1}^{n} \frac{\partial\alpha}{\partial z_j} dz_j \quad \text{and} \quad \bar{\partial}\alpha := \sum_{j=1}^{n} \frac{\partial\alpha}{\partial \bar{z}_j} d\bar{z}_j.$$

Note that

$$d = \partial + \overline{\partial},$$

i.e. for each for each smooth function α, we may write

$$d\alpha = \partial\alpha + \overline{\partial}\alpha,$$

where $\partial\alpha \in C^\infty(\Omega_X^{1,0})$ and $\overline{\partial}\alpha \in C^\infty(\Omega_X^{0,1})$. More in general, if $\omega \in C^\infty(\Omega_X^{p,q})$, then we can define

$$\partial\omega := \sum_{|I|=p,|J|=q} \partial(\alpha_{I,J}) \wedge dz_I \wedge d\bar{z}_J \qquad \overline{\partial}\omega := \sum_{|I|=p,|J|=q} \overline{\partial}(\alpha_{I,J}) \wedge dz_I \wedge d\bar{z}_J$$

and, we have

$$d\omega = \partial\omega + \overline{\partial}\omega,$$

i.e. also in this case we may write $d = \partial + \overline{\partial}$. Note that $\partial\omega \in C^\infty(\Omega_X^{p+1,q})$ and $\overline{\partial}\omega \in C^\infty(\Omega_X^{p,q+1})$. By (4.35) and by linearity, we have that $\partial, \overline{\partial}$ can be extended as linear maps

$$\partial, \overline{\partial} \colon C^\infty(X, \Omega_{X,\mathbb{C}}^k) \to C^\infty(X, \Omega_{X,\mathbb{C}}^{k+1}).$$

Lemma 4.37 (Properties of ∂ and $\overline{\partial}$). *Let X be a complex manifold.*

1. *The maps d, ∂, and $\overline{\partial}$ map between the following spaces:*

$$\begin{aligned} \mathrm{d} = \partial + \overline{\partial} : C^\infty(X, \Omega_{X,\mathbb{C}}^k) &\to C^\infty(X, \Omega_{X,\mathbb{C}}^{k+1}), \\ \partial : C^\infty(X, \Omega_X^{p,q}) &\to C^\infty(X, \Omega_X^{p+1,q}), \ \textit{and} \\ \overline{\partial} : C^\infty(X, \Omega_X^{p,q}) &\to C^\infty(X, \Omega_X^{p,q+1}). \end{aligned}$$

2. *(Leibniz rule) Let $\omega \in C^\infty(X, \Omega_{X,\mathbb{C}}^k)$ and $\eta \in C^\infty(X, \Omega_{X,\mathbb{C}}^\ell)$.*
   Then
   $$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^k \omega \wedge \partial\eta$$
   and
   $$\overline{\partial}(\omega \wedge \eta) = \overline{\partial}\omega \wedge \eta + (-1)^k \omega \wedge \overline{\partial}\eta.$$

3. *The following hold:*
   $$\partial^2 = 0, \quad \bar{\partial}\partial + \partial\bar{\partial} = 0, \quad \overline{\partial}^2 = 0.$$

Proof.

1. This follows directly from the definitions.

2. By linearity, we may assume that ω has type (p, q) and η has type (p', q'), where $k = p + q$ and $\ell = p' + q'$. By the Leibnitz rule for d, we have
   $$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

Thus,

$$\begin{aligned}(\partial+\bar{\partial})(\omega\wedge\eta) &= d(\omega\wedge\eta)\\ &= d\omega\wedge\eta+(-1)^k\omega\wedge d\eta\\ &= \partial\omega\wedge\eta+\bar{\partial}\omega\wedge\eta+(-1)^k\omega\wedge\partial\eta+(-1)^k\omega\wedge\bar{\partial}\eta\\ &= \left(\partial\omega\wedge\eta+(-1)^k\omega\wedge\partial\eta\right)+\left(\bar{\partial}\omega\wedge\eta+(-1)^k\omega\wedge\bar{\partial}\eta\right)\end{aligned}$$

gives a decomposition in forms of type $(p+p'+1, q+q')$ and $(p+p', q+q'+1)$ respectively. Thus, the Lemma follows.

3. By linearity, it is enough to check the equalities on a form ω of type (p,q). Since $d\circ d=0$, we have

$$\begin{aligned}0 = d^2\omega &= (\partial+\bar{\partial})^2\omega\\ &= \partial^2\omega+(\bar{\partial}\partial+\partial\bar{\partial})\omega+\bar{\partial}^2\omega.\end{aligned}$$

Since $\partial^2\omega$, $(\bar{\partial}\partial+\partial\bar{\partial})\omega$ and $\bar{\partial}^2\omega$ have different type, i.e. type $(p+2,q)$, $(p+1,q+1)$ and $(p,q+2)$ respectively, they must all vanish.

□

Definition 4.38. For each pair (p,q), we define

$$\begin{aligned}\mathcal{Z}^{p,q}(X) :&= \operatorname{Ker}\left(\bar{\partial}\colon C^\infty(X,\Omega_X^{p,q})\to C^\infty(X,\Omega_X^{p,q+1})\right)\\ &= \{\omega\in C^\infty(X,\Omega_X^{p,q}) \mid \bar{\partial}\omega=0\}\end{aligned}$$

and, for each $q\geq 1$, we define

$$\begin{aligned}\mathcal{B}^{p,q}(X) :&= \operatorname{Im}\left(\bar{\partial}\colon C^\infty(X,\Omega_X^{p,q-1})\to C^\infty(X,\Omega_X^{p,q})\right)\\ &= \{\omega\in C^\infty(X,\Omega_X^{p,q}) \mid \omega=\bar{\partial}\eta \text{ for some } \eta\in C^\infty(X,\Omega_X^{p,q-1})\}\end{aligned}$$

For convenience, we define $\mathcal{B}^{p,0}=0$ for each p.
The standard terminology is to say that the forms in $\mathcal{Z}^{p,q}$ are **$\bar{\partial}$-closed** and the forms in $\mathcal{B}^{p,q}$ are **$\bar{\partial}$-exact**.
By the previous Lemma, it follows that

$$\mathcal{B}^{p,q}(X)\subset\mathcal{Z}^{p,q}(X)$$

for each p,q.
Thus, we may define

$$H^{p,q}(X) := \mathcal{Z}^{p,q}(X)/\mathcal{B}^{p,q}(X).$$

The group $H^{p,q}(X)$ is called **Dolbeault cohomology group** of X. Each $H^{p,q}(X)$ is manifestly a $\mathbb{C}$-vector space. If it is finite dimensional, then its dimension

$$h^{p,q}(X) := \dim H^{p,q}(X)$$

is called a **Hodge number** of X.

We first study the groups $H^{p,0}(X)$.

Proposition 4.39. *Let X be a complex manifold and let $p\geq 0$ then we have an isomorphism*

$$H^{p,0}(X)\simeq H^0(X,\Omega_X^p),$$

where the notation on the right hand side stands for the space of holomorphic sections of Ω_X^p, as in Equation 4.16. In particular, if X is compact, then $H^{0,0}(X)=\mathbb{C}$.

Proof. We have
$$H^{p,0}(X) = \mathcal{Z}^{p,0}(X) = \{\omega \in C^\infty(X, \Omega_X^{p,0}) \mid \overline{\partial}\omega = 0\}.$$
Let $\omega \in C^\infty(\Omega_X^{p,0})$ such that $\overline{\partial}\omega = 0$. Locally, we may write
$$\omega = \sum_{|I|=p} \alpha_I dz_I,$$
where α_I is a smooth function for each I. Then
$$0 = \overline{\partial}\alpha = \sum_{|I|=p} \frac{\partial}{\partial \overline{z}_j} \alpha_I d\overline{z}_j \wedge dz_I.$$
Since, locally the forms $d\overline{z}_j \wedge dz_I$ are linearly independent, it follows that
$$\frac{\partial}{\partial \overline{z}_j}\alpha_I = 0 \qquad \text{for every } I, j.$$
Thus, α_I is holomorphic for every I. In particular, ω is a holomorphic section of Ω_X^p. Similarly, if ω is a holomorphic section of Ω_X^p then $[\omega] \in H^{p,0}(X)$.
The last part of the Proposition follows from Theorem 3.7. □

Recall that for a ball $\mathbb{B}$ in $\mathbb{R}^n$ (or possibly infinite radius), de Rham cohomology $H^k(\mathbb{B})$ vanish for $k \geq 1$. The following is a complex analogue of this fact (for the proof, see Corollary 1.3.9 in Huybrecht's textbook)

Lemma 4.40 ($\overline{\partial}$-Poincaré lemma). *Let $\mathbb{D}$ be a polydisc $D(c, r) \subset \mathbb{C}^n$, where some (or all) of the components of the polyradius $r = (r_1, ..., r_n)$ are allowed to be infinite.*
Then $H^{p,q}(\mathbb{D}) = 0$, for $p \geq 0, q \geq 1$.

Proof. [7, Corollary 1.3.9]. □

The following is a complex analogue of Poincarè duality:

Theorem 4.41. *(Serre duality) Let X be a compact complex manifold of dimension n. Then for all p, q, we have $H^{p,q}(X)^* = H^{n-p,n-q}(X)$. In particular, the Hodge numbers $h^{p,q}(X)$ and $h^{n-p,n-q}(X)$ are equal.*

Even though it is a statement about complex manifolds, its proof uses techniques from Kähler geometry involving a choice of Riemannian structure on the underlying manifold. We will cover Kähler manifolds later, but we may not get as far as to prove this theorem.

4.4 $\overline{\partial}$ operator on a holomorphic vector bundle

Definition 4.42. Let E be a holomorphic vector bundle over a complex manifold X. For each $p, q \geq 0$, we consider the complex bundle $\Omega_X^{p,q}(E) := \Omega_X^{p,q} \otimes E$. The sections of $\Omega_X^{p,q} \otimes E$ are called E-valued (p, q)-forms.

Proposition 4.43. *Let E be a holomorphic vector bundle over a complex manifold X. Then there exists a $\mathbb{C}$-linear operator*
$$\bar{\partial}_E : C^\infty(X, \Omega_X^{p,q}(E)) \to C^\infty(X, \Omega_X^{p,q+1}(E))$$
with $\bar{\partial}_E^2 = 0$ that satisfies the Leibniz rule
$$\bar{\partial}_E(f\alpha) = \bar{\partial}(f) \wedge \alpha + f\bar{\partial}_E(\alpha),$$
for any C^∞-function $f : X \to \mathbb{C}$ and $\alpha \in C^\infty(X, \Omega_X^{p,q}(E))$.

Proof. Let $s_1, \ldots, s_r$ be a (holomorphic) local frame of E defined over an open set $U \subset X$, as in Equation 4.19. Then any section $\alpha \in C^\infty(X, \Omega_X^{p,q}(E))$ locally has the expression $\alpha = \sum_{j=1}^r \alpha_j \otimes s_j$ with $\alpha_j \in C^\infty(U, \Omega_X^{p,q})$. We define

$$\bar{\partial}_E(\alpha) := \sum_{j=1}^r \bar{\partial}(\alpha_j) \otimes s_j.$$

We claim that this definition does not depend on the choice of the local frame. Let $s'_1, \ldots s'_r$ be a different holomorphic local frame of E over U, and $\bar{\partial}'_E$ be the corresponding operator. Then we can express $s_j = \sum_{i=1}^r \psi_{ij} s'_i$, $1 \le i, j \le r$, where all ψ_{ij} are holomorphic functions $U \to \mathbb{C}$. Then

$$\sum_{j=1}^r \alpha_j \otimes s_j = \sum_{j=1}^r \alpha_j \otimes \left(\sum_{i=1}^r \psi_{ij} s'_i\right) = \sum_{i=1}^r \left(\sum_{j=1}^r \psi_{ij}\alpha_j\right) \otimes s'_i.$$

Using the Leibniz rule for $\bar{\partial}$ (Equation 4.37), we get $\bar{\partial}(\psi_{ij}\alpha_j) = \psi_{ij}\bar{\partial}(\alpha_j)$. Therefore,

$$\bar{\partial}'_E(\alpha) = \sum_{i=1}^r \bar{\partial}\left(\sum_{j=1}^r \psi_{ij}\alpha_j\right) \otimes s'_i = \sum_{i=1}^r \left(\sum_{j=1}^r \psi_{ij}\bar{\partial}(\alpha_j)\right) \otimes s'_i =$$

$$= \sum_{j=1}^r \bar{\partial}(\alpha_j) \otimes \left(\sum_{i=1}^r \psi_{ij} s'_i\right) = \sum_{j=1}^r \bar{\partial}(\alpha_j) \otimes s_j = \bar{\partial}_E(\alpha).$$

This proves that $\bar{\partial}_E$ is well-defined. Checking $\bar{\partial}_E^2 = 0$ and the Leibniz rule for $\bar{\partial}_E$ is left as an exercise. □

Exercise 4.44. *Prove that the operator $\overline{\partial}_E$ constructed in the proof of Equation 4.43 satisfies the following conditions*

1. *$\overline{\partial}_E(s) = 0$, for every locally defined holomorphic section s of E, and*
2. *(more general version of the Leibniz rule)*

$$\overline{\partial}_E(\alpha \wedge \beta) = \overline{\partial}(\alpha) \wedge \beta + (-1)^{p+q}\alpha \wedge \overline{\partial}_E(\beta),$$

for $\alpha \in C^\infty(X, \Omega_X^{p,q})$, $\beta \in C^\infty(X, \Omega_X^{r,s}(E))$.

Prove that there is a unique $\overline{\partial}_E$ as in Equation 4.43 satisfying the additional conditions 1)-2).

4.5 Dolbeault cohomology of a holomorphic vector bundle

Definition 4.45. Let E be a holomorphic vector bundle over a complex manifold X. Let $\overline{\partial}_E$ be the operator consructed in the proof of Equation 4.43. For each $q \ge 0$, we define

$$H^q(X,E) := \frac{\operatorname{Ker}\left(\overline{\partial}_E : C^\infty(X, \Omega_X^{0,q}(E)) \to C^\infty(X, \Omega_X^{0,q+1}(E))\right)}{\operatorname{Im}\left(\overline{\partial}_E : C^\infty(X, \Omega_X^{0,q-1}(E)) \to C^\infty(X, \Omega_X^{0,q}(E))\right)} =: \frac{Z^q(X,E)}{B^q(X,E)}.$$

The groups $H^q(X,E)$ are called **Dolbeault cohomology groups** of E. Note that previously defined Dolbeault cohomology groups of X are recovered by $H^{p,q}(X) = H^q(X, \Omega_X^p)$. By $h^q(X,E)$ we denote the complex dimension of $H^q(X,E)$ viewed as a $\mathbb{C}$-vector space.

Let n be the complex dimension of X. Then $\Omega_X^{0,q}$ is the zero vector bundle, if $q > n$. Therefore, $H^q(X,E) = 0$ for $q > n$.

Remark 4.46. Note that each $H^q(X,-)$ is a covariant functor, in the sense that for a holomorphic bundle morphism $\alpha : E \to F$, there is a $\mathbb{C}$-linear map $H^q(X,E) \to H^q(X,F)$. The latter is induced by applying $1\otimes\alpha$ to a element of $C^\infty(X, \Omega_X^{0,q} \otimes E)$. The fact that α is holomorphic implies that $1 \otimes \alpha$ sends $\overline{\partial}_E$-closed (resp. $\overline{\partial}_E$-exact) E-valued $(0,q)$-forms to $\overline{\partial}_F$-closed (resp. $\overline{\partial}_E$-exact) F-valued $(0,q)$-forms.

The following fact can be deduced directly from the definitions.

Lemma 4.47. *For two holomorphic vector bunldes E and G over X, we have*

$$H^q(X, E\oplus G) = H^q(X,E) \oplus H^q(X,G), \quad \textit{for all } q \geq 0.$$

Let us discuss an important generalization of Lemma 4.47, which is serves as a major computational tool.

Definition 4.48. Let E, F, G be holomorphic vector bundles over X. We say that the sequence of holomorphic bundle morphisms

$$0 \longrightarrow E \xrightarrow{\alpha} F \xrightarrow{\beta} G \longrightarrow 0$$

form a short exact sequence, if for each $x \in X$, we have the short exact sequence of $\mathbb{C}$-vector spaces

$$0 \longrightarrow E(x) \xrightarrow{\alpha(x)} F(x) \xrightarrow{\beta(x)} G(x) \longrightarrow 0,$$

that is, $\alpha(x)$ is injective, $\beta(x)$ is surjective, and $\mathrm{Im}(\alpha(x)) = \mathrm{Ker}(\beta(x))$.

Proposition 4.49 (Long exact cohomological sequence). *Let*

$$0 \longrightarrow E \xrightarrow{\alpha} F \xrightarrow{\beta} G \longrightarrow 0$$

be a short exact sequence of holomorphic vector bundles over a complex manifold X. Then there is an exact sequence of $\mathbb{C}$-vector spaces of the form

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,E) \arrow[r] & H^0(X,F) \arrow[r] & H^0(X,G) \arrow[dll] \\ & H^1(X,E) \arrow[r] & H^1(X,F) \arrow[r] & H^1(X,G) \arrow[dll] \\ & H^2(X,E) \arrow[r] & \cdots & \end{tikzcd} \qquad (*)$$

Recall that exactness of a sequence of vector spaces means that at each position the image of incoming morphism equals the kernel of the outgoing one. In order to prove this theorem, we will use the *Snake Lemma*:

Proposition 4.50 (Snake Lemma). *Consider the following commutative diagram of vector bundles where the rows are exact sequences:*

$$\begin{tikzcd} & E \arrow[r, "\alpha"] \arrow[d, "d_E"] & F \arrow[r, "\beta"] \arrow[d, dashed, "d_F"] & G \arrow[r] \arrow[d, dashed, "d_g"] & 0 \\ 0 \arrow[r] & E' \arrow[r, "\alpha'"] & F' \arrow[r, "\beta'"] & G' & \end{tikzcd}$$

Then there exists $\delta : \mathrm{Ker}(d_G) \to \mathrm{CoKer}(d_E)$ *fitting into the following exact sequence:*

$$\begin{tikzcd}
& \ker d_E \arrow[r] \arrow[d] & \ker d_F \arrow[r] \arrow[d] & \ker d_G \arrow[d] \arrow[dddll, rounded corners, "\delta"] & \\
& E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E' \arrow[r] \arrow[d] & F' \arrow[r] \arrow[d] & G' \arrow[d] & \\
& \mathrm{CoKer}\, d_E \arrow[r] & \mathrm{CoKer}\, d_F \arrow[r] & \mathrm{CoKer}\, d_G &
\end{tikzcd}$$

Proof. **Construction of** δ: Let $\tau \in \ker d_G$. Because β is surjective, there exists $\sigma \in F$ such that $\beta(\sigma) = \tau$. We have $\beta'(d_F(\sigma)) = d_G(\beta(\sigma)) = d_G(\tau) = 0$. Because of exactness at F' in the original diagram there exists $\rho \in E'$ such that $\alpha'(\rho) = d_F(\sigma)$. As α is injective, we have that ρ is unique. We define $\delta(\tau) = [\rho] \in \mathrm{CoKer}\, d_E$.
Check that δ **is well-defined**: in the construction we could have chosen $\sigma' \in F$ instead of σ. Let $\alpha'(\rho') = d_F(\sigma')$. Then $\beta(\sigma - \sigma') = 0$, so $\sigma - \sigma' = \alpha(\kappa)$ for some $\kappa \in E$. Thus, $\alpha(\rho - \rho') = d_F(\sigma - \sigma') = d_F(\alpha(\kappa)) = \alpha'(d_E(\kappa))$. By injectivity of α', this means $\rho - \rho' = d_E(\kappa)$, i.e. $[\rho] = [\rho'] \in \mathrm{CoKer}(d_E)$.
Exactness checks: we must check that $\mathrm{Im}\,\beta(\ker d_F \to \ker d_G) = \mathrm{Ker}\,\delta$ and $\mathrm{Im}\,\delta = \mathrm{Ker}\,\alpha'$. We check that $\mathrm{Ker}\,\delta \subset \mathrm{Im}\,\beta$, the other checks are analogous. Let $\delta(\tau) = [\rho] = 0 \in \mathrm{CoKer}\, d_E$, i.e. $\rho = d_E(\xi) \in E'$ for some $\xi \in E$. With the notation from the construction of δ, we have that $\beta(\sigma - \alpha(\xi)) = \beta(\sigma) = \tau$, so we know that $\tau \in \mathrm{Im}\,\beta(F \to G)$, but it remains to check that the preimage $\sigma - \alpha(\xi)$ is actually in $\ker d_F$, not just in F. That is the case, because $d_F(\sigma) = \alpha'(\rho) = \alpha'(d_E(\xi)) = d_F(\alpha(\xi))$, therefore $d_F(\sigma - \alpha(\xi)) = 0$. □

We are now ready to prove Proposition 4.49:

Proof of Proposition 4.49. The horizontal arrows in $(*)$ of Proposition 4.49 were defined in Remark 4.46. For the diagonal arrows: from the Snake Lemma we get:

$$\begin{tikzcd}
 & \ker\overline{\partial} \arrow[r] \arrow[d] & \ker\overline{\partial} \arrow[r] \arrow[d] & \ker\overline{\partial} \arrow[d] \arrow[dddll, "\delta_q"] & \\
 & C^\infty(X,\Omega_X^{0,q}(E)) \arrow[r] \arrow[d, "\overline{\partial}"] & C^\infty(X,\Omega_X^{0,q}(F)) \arrow[r] \arrow[d, "\overline{\partial}"] & C^\infty(X,\Omega_X^{0,q}(G)) \arrow[r] \arrow[d, "\overline{\partial}"] & 0 \\
0 \arrow[r] & C^\infty(X,\Omega_X^{0,q+1}(E)) \arrow[r] \arrow[d] & C^\infty(X,\Omega_X^{0,q+1}(F)) \arrow[r] \arrow[d] & C^\infty(X,\Omega_X^{0,q+1}(G)) \arrow[d] & \\
 & \operatorname{CoKer}\overline{\partial} \arrow[r] & \operatorname{CoKer}\overline{\partial} \arrow[r] & \operatorname{CoKer}\overline{\partial} &
\end{tikzcd}$$

For this to be a well defined map $\delta_q : H^q(X,G) \to H^{q+1}(X,E)$ it remains to check that

1. $\delta_q(B^q(X,G)) = 0$, and
2. $\overline{\partial}(\delta_q(\tau)) = 0$ for all $\tau \in Z^q(X,G)$.

For point 1., let $\tau = \overline{\partial}\xi \in C^\infty(X,\Omega_X^{0,q}(G))$. Let η such that $\beta(\eta) = \xi$. Then $\beta(\overline{\partial}\eta) = \overline{\partial}(\beta(\eta)) = \overline{\partial}(\xi) = \tau$. Therefore $\delta_q(\tau) = \delta_q(\overline{\partial}(\xi)) \in \alpha^{-1}(\overline{\partial\partial}\eta) = \alpha^{-1}(0)$ by the definition of δ_q from the proof of Proposition 4.50. But α is injective, so $\delta_q(\tau) = 0$.
For point 2., we have with the notation from the proof of Proposition 4.50 for $\delta_q(\tau) = [\rho]$: $\alpha(\overline{\partial}\rho) = \overline{\partial}(\alpha(\rho)) = \overline{\partial\partial}(\sigma) = 0$. As α is injective, we have that $\overline{\partial}\rho = 0$. □

Note that in the proof of the existence of the long exact sequence, the only property of the ∂-operator we used was that $\overline{\partial\partial} = 0$. This is the reason why in every cohomology (or homology) theory one gets a long exact sequence in cohomology (or homology) from a short exact sequence.
Serre duality (Theorem 4.41) generalizes to Dolbeault cohomology of holomorphic vector bundles as follows:

Theorem 4.51. *(Serre duality) Let X be a compact complex manifold of dimension n. Let E be a holomorphic vector bundle over X. Then for all $0 \le q \le n$ we have*

$$H^q(X,E)^* = H^{n-q}(X,E^* \otimes K_X),$$

where $K_X = \Omega_X^n$ is the canonical line bundle of X.

4.6 de Rham Cohomology

Similarly as in Section 4.3, given a complex manifold X of dimension n, we define, for each $k \geq 0$,

$$\begin{aligned}\mathcal{Z}^k(X) &:= \operatorname{Ker}\left(d\colon C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}})\right)\\ &= \{\omega \in C^\infty(X, \Omega^k_{X,\mathbb{C}}) \mid d\omega = 0\}\end{aligned}$$

and, for each $k \geq 1$, we define

$$\begin{aligned}\mathcal{B}^k(X) &:= \operatorname{Im}\left(d\colon C^\infty(X, \Omega^{k-1}_{X,\mathbb{C}}) \to C^\infty(X, \Omega^k_{X,\mathbb{C}})\right)\\ &= \{\omega \in C^\infty(X, \Omega^k_{X,\mathbb{C}}) \mid \omega = d\eta \text{ for some } \eta \in C^\infty(X, \Omega^{k-1}_{X,\mathbb{C}})\}\end{aligned}$$

For convenience, we define $\mathcal{B}^0 = 0$.

Since $d \circ d = 0$, it follows that

$$\mathcal{B}^k(X) \subset \mathcal{Z}^k(X)$$

for each $k \geq 0$.
Thus, we may define

$$H^k(X, \mathbb{C}) := \mathcal{Z}^k(X)/\mathcal{B}^k(X).$$

The group $H^k(X)$ is called the **de Rham cohomology group** of X. If it is finite dimensional, then their dimension

$$b_k(X) := \dim H^k(X, \mathbb{C})$$

is called **Betti number** of X. The **Euler characteristic** of X is defined as

$$\chi(X) = \sum_{k=0}^{2n} (-1)^k b_k.$$

Similarly, by considering only real forms, i.e.

$$\begin{aligned}\mathcal{Z}^k_{\mathbb{R}}(X) &:= \operatorname{Ker}\left(d\colon C^\infty(X, \Omega^k_{X,\mathbb{R}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{R}})\right)\\ &= \{\omega \in C^\infty(X, \Omega^k_{X,\mathbb{R}}) \mid d\omega = 0\}\end{aligned}$$

and, for each $k \geq 1$, we define

$$\begin{aligned}\mathcal{B}^k_{\mathbb{R}}(X) &:= \operatorname{Im}\left(d\colon C^\infty(X, \Omega^{k-1}_{X,\mathbb{R}}) \to C^\infty(X, \Omega^k_{X,\mathbb{R}})\right)\\ &= \{\omega \in C^\infty(X, \Omega^k_{X,\mathbb{R}}) \mid \omega = d\eta \text{ for some } \eta \in C^\infty(X, \Omega^{k-1}_{X,\mathbb{R}})\}\end{aligned}$$

then we can define

$$H^k(X, \mathbb{R}) := \mathcal{Z}^k_{\mathbb{R}}(X)/\mathcal{B}^k_{\mathbb{R}}(X).$$

Remark 4.52. If X and X' are diffeomorphic complex manifolds then $H^k(X, \mathbb{C}) \simeq H^k(X', \mathbb{C})$ for any $k \geq 0$. Moreover, if X is a complex manifold, then for any $k \geq 0$, we have

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

5 Connections, curvature and metrics

5.1 Connections

Definition 5.1. Let $E \to X$ be a complex bundle. A *connection* on E is a $\mathbb{C}$-linear map

$$\nabla : C^\infty(X, E) \to C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E)$$

satisfying the Leibniz rule

$$\nabla(fs) = \mathrm{d}f \otimes s + f \cdot \nabla s$$

for any local function f on X and any local section s of E.

Definition 5.2. A section $s \in C^\infty(X, E)$ is called *parallel* if $\nabla s = 0$.

Proposition 5.3. *Let ∇, ∇' be connections on $E \to X$. Then $\nabla - \nabla'$ is $C^\infty(X, \mathbb{C})$-linear and therefore defines an element in $C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes \operatorname{End} E)$. Here, $C^\infty(X, \mathbb{C})$-linear means that*

$$(\nabla - \nabla')(f \cdot s) = f \cdot (\nabla - \nabla')(s)$$

for all $s \in C^\infty(X, E)$ and $f : X \to \mathbb{C}$ smooth.
Conversely, if ∇ is a connection and $a \in C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes \operatorname{End} E)$, then $\nabla + a$ is again a connection on E.

Proof. The property $(\nabla - \nabla')(fs) = f(\nabla - \nabla')s$ follows from the Leibniz rule for ∇ and ∇'. Thus, for $V \in T_x X$ the map

$$\begin{aligned} \nabla_V - \nabla'_V : E_x &\to E_x \\ s(x) &\mapsto ((\nabla_V - \nabla'_V)s)(x) \text{ for a local section } s \text{ of } E \end{aligned}$$

is well-defined, i.e. independent of the choice of s. That is, $(\nabla - \nabla') \in C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes \operatorname{End} E)$.
For the second part observe that

$$(\nabla + a)(fs) = \nabla(fs) + a(fs) = \mathrm{d}f \otimes s + f\nabla s + fa(s) = \mathrm{d}f \otimes s + f((\nabla + a)s),$$

i.e. $\nabla + a$ satisfies the Leibniz rule and is therefore again a connection. $\square$

Corollary 5.4. *The set of all connections on a complex vector bundle $E \to X$ is in a natural way an affine space over the infinite-dimensional complex vector space $C^\infty(X, \Omega^1_{C,C} \otimes \operatorname{End} E)$.*

Remark 5.5 (Local calculations). On the trivial bundle $E = X \times \mathbb{C}^r$ we can define the *trivial connection*

$$\mathrm{d} : C^\infty(X, \underline{\mathbb{C}^r}) \to C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes \underline{\mathbb{C}^r}).$$

By the previous corollary, every other connection is of the form $\nabla = \mathrm{d} + A$ for $A \in C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes \operatorname{End} E) = C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes \underline{\mathbb{C}^{r\times r}})$, where A is called a *matrix-valued* 1*-form.*
For a general bundle $E \to X$ with connection ∇, let $\psi : E|_U \to U \times \mathbb{C}^r$ be a trivialisation. We can then write $\nabla = \psi^{-1} \circ (\mathrm{d} + A) \circ \psi$. People often use the shorthand notation "$\nabla = \mathrm{d} + A$". Here, $A \in C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes \underline{\mathbb{C}^{r\times r}})$ depends on the choice of trivialisation ψ. For example, if $\phi : U \to \mathrm{GL}(r, \mathbb{C})$, so that ψ' is a second trivialisation satisfying $\phi \cdot \psi' = \psi$, we have:

$$\nabla = (\psi')^{-1} \circ (\mathrm{d} + A') \circ \psi'$$

for $A' = \phi^{-1}\,\mathrm{d}\phi + \phi^{-1}A\phi$. This holds because

$$\begin{aligned} \psi^{-1}(\mathrm{d} + A)\psi &= \psi^{-1}\phi\phi^{-1}(\mathrm{d} + A)\phi\phi^{-1}\psi \\ &= (\psi')^{-1}(\phi^{-1}(\mathrm{d} + a)\phi\psi' \\ &= (\psi')^{-1}(\phi^{-1}(\mathrm{d}\phi) + \underbrace{\phi^{-1}\phi}_{=\mathrm{Id}}\,\mathrm{d} + \phi^{-1}A\phi)\psi', \end{aligned}$$

where we used the product rule for matrix valued functions

$$\mathrm{d}\left(\phi\cdot s\right)=\left(\mathrm{d}\phi\right)\cdot s+\phi\cdot\left(\mathrm{d}s\right)\text{ for }s\in C^{\infty}(U,\underline{\mathbb{C}^{r}})$$

in the last step.
Furthermore, for $x_0 \in M$ one can always choose ψ' such that $A'(x_0) = 0$. For this, one can choose

$$\phi(x) := \mathrm{Id} - \sum_{i=1}^{n} x_i A_i(0) + \text{h.o.t.},$$

where $(x_1, \ldots, x_n)$ are local coordinates around x_0 which have x_0 as the origin, and the A_i are given by

$$A = \sum_{i=1}^{n} A_i \,\mathrm{d}x_i.$$

Definition 5.6. Let $E \to X$ be a complex vector bundle. We define an extension of ∇ to a map

$$\nabla : C^{\infty}(X, \Omega^{k}_{X,\mathbb{C}} \otimes E) \to C^{\infty}(X, \Omega^{k+1}_{X,\mathbb{C}} \otimes E)$$

by requiring that it satisfy the Leibniz rule

$$\nabla(\alpha \otimes s) := (\mathrm{d}\alpha) \otimes s + (-1)^{k} \alpha \wedge \nabla s$$

for a local k-form α and a local section s. Check that then also the generalised Leibniz rule holds for $t \in C^{\infty}(X, \Omega^{k}_{X,\mathbb{C}} \otimes E)$ and $\beta \in C^{\infty}(X, \Omega^{k}_{X,\mathbb{C}})$:

$$(\nabla\beta \wedge t) = \mathrm{d}(\beta) \wedge t + (-1)^{k} \beta \wedge \nabla t.$$

Definition 5.7. The *curvature* F_{∇} of a connection ∇ on a complex vector bundle $E \to X$ is defined as

$$F_{\nabla} := \nabla \circ \nabla : C^{\infty}(X, \Omega^{0}_{X,\mathbb{C}} \otimes E) \to C^{\infty}(X, \Omega^{2}_{X,\mathbb{C}} \otimes E).$$

Lemma 5.8. *The curvature F_{∇} is $C^{\infty}(X, \mathcal{O}_X)$-linear, i.e. $F_{\nabla} \in C^{\infty}(X, \Omega^{2}_{X,\mathbb{C}} \otimes \mathrm{End}(E))$.*

Proof. To check linearity, note that

$$F_{\nabla}(fs) = \nabla(\nabla(fs)) = \nabla(\mathrm{d}f \otimes s + f\nabla s) = \mathrm{d}\,\mathrm{d}f \otimes s - \mathrm{d}f \wedge \nabla s + \mathrm{d}f \wedge \nabla s + f\nabla\nabla s.$$

One sees as in Proposition 5.3 that this implies $F_{\nabla} \in C^{\infty}(X, \Omega^{2}_{X,\mathbb{C}} \otimes \mathrm{End}(E))$. ◻

The following lemma gives an alternative formula for the curvature. The curvature of the Levi-Civita connection of a Riemannian metric is usually written in this way.

Lemma 5.9. *Let $E \to X$ be a complex vector bundle. Let $V, W \in T_xX$ and $\xi \in E_x$. Let $\widetilde{V}, \widetilde{W} \in \mathfrak{X}(X)$ extensions of V and W, i.e. vector fields satisfying $\widetilde{V}(x) = V$ and $\widetilde{W}(x) = W$. Let $s \in C^{\infty}(X, E)$ be an extension of ξ, i.e. $s(x) = \xi$. Then*

$$F_{\nabla}(V, W)\xi = \left(\nabla_{\widetilde{V}}\nabla_{\widetilde{W}}s - \nabla_{\widetilde{W}}\nabla_{\widetilde{V}}s - \nabla_{[\widetilde{V},\widetilde{W}]}s\right)(x). \qquad (*)$$

Proof. By Lemma 5.8, the left hand side of equation $(*)$ is $C^{\infty}(X, \mathcal{O}_X)$-linear by Lemma 5.8. One checks that the right hand side of equation $(*)$ is also $C^{\infty}(X, \mathcal{O}_X)$-linear. Because of this, it suffices to check the equality for $\widetilde{V} = \frac{\partial}{\partial x_i}$ and $\widetilde{W} = \frac{\partial}{\partial x_j}$, where $(x_1, \ldots, x_n)$ are local coordinates on X.

Then

$$\begin{aligned}
F_\nabla(V,W)\xi &= \left(\nabla\left(\sum_k \mathrm{d}x_k \otimes \nabla_{\frac{\partial}{\partial x_k}} s\right)\right)(\widetilde{V},\widetilde{W})(x)\\
&= \left(-\sum_{k,l} \mathrm{d}x_k \wedge \mathrm{d}x_l \otimes \nabla_{\frac{\partial}{\partial x_l}} \nabla_{\frac{\partial}{\partial x_k}} s\right)(\widetilde{V},\widetilde{W})(x)\\
&= \left(\nabla_{\widetilde{V}}\nabla_{\widetilde{W}} s - \nabla_{\widetilde{W}}\nabla_{\widetilde{V}} s\right)(x)\\
&= \left(\nabla_{\widetilde{V}}\nabla_{\widetilde{W}} s - \nabla_{\widetilde{W}}\nabla_{\widetilde{V}} s - \underbrace{\nabla_{[\widetilde{V},\widetilde{W}]} s}_{=0}\right)(x),
\end{aligned}$$

where in the second step we used the Leibniz rule and $\mathrm{d}\,\mathrm{d}x_k = 0$; in the third step we used

$$\mathrm{d}x_k \wedge \mathrm{d}x_l \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \delta_{ki}\delta lj - \delta_{li}\delta_{kj};$$

and in the last step we used that canonical basis vector fields commute, see Proposition 4.29 point 7. □

Proposition 5.10.

1. *Let $\nabla = \mathrm{d} + A$ be a connection on the trivial bundle $E = M \times \mathbb{C}^r$. Then $F_\nabla = \mathrm{d}A + A \wedge A$, where $A \wedge A \in C^\infty(M, \Omega^2_{M,\mathbb{C}} \otimes \operatorname{End} E)$ is defined by $(A \wedge A)(V,W) = A(V) \circ A(W) - A(W) \circ A(V)$, where $\circ$ denotes composition of endomorphisms.*

2. *Let $\psi, \psi; E\ |_U \longrightarrow U \times \mathbb{C}^r$ be two local trivialisations of a bundle E such that $\phi \cdot \psi' = \psi$. Let $\nabla = \psi^{-1}(\mathrm{d} + A)\psi = (\psi')^{-1}(\mathrm{d} + A')\psi'$ be the local formulae for ∇. Then*

$$\mathrm{d}A' + A' \wedge A' = \phi^{-1}(\mathrm{d}A + A \wedge A)\phi.$$

3. *Let ∇ be a connection on E and $a \in C^\infty(M, \Omega^1_{M,\mathbb{C}} \otimes \operatorname{End} E)$, then*

$$F_{\nabla + a} = F_\nabla + \nabla(a) + a \wedge a.$$

4. *(Bianchi identity) $\nabla(F_\nabla) = 0 \in C^\infty(M, \Omega^3_{M,\mathbb{C}} \otimes \operatorname{End} E)$.*

Proof.

1. We have

$$F_\nabla(s) = (\mathrm{d} + A)(\mathrm{d} + A)s = \underbrace{\mathrm{d}\,\mathrm{d}s}_{=0} + \mathrm{d}(As) + A \wedge (\mathrm{d}s) + (A \wedge A)s = (\mathrm{d}A)(s) + (A \wedge A)s.$$

   For the second inequality we used the Leibniz rule from Definition 5.6 and for the last inequality we used the Leibniz rule $\mathrm{d}\,(As) = (\mathrm{d}A)(s) - A \wedge \mathrm{d}s$ for d.

2. This is left as an exercise.

3. We have

$$F_{\nabla + a}(s) = (\nabla + a)(\nabla + a)s = \nabla\nabla s + a \wedge as + \underbrace{\nabla(as)}_{(\nabla a)s - a(\nabla s)} + a(\nabla s) = F_\nabla(s) + a \wedge as + (\nabla a)s,$$

   where $\nabla(as) = (\nabla a)s - a(\nabla s)$ follows from writing $s = \sum \alpha_i s_i$ in a local frame, and $as = \sum_{a_{ij}} \alpha_j s_i$, and then using the Leibniz rule for d in this frame.

4. This is also left as an exercise. □

5.2 Hermitian metrics

Recall the following definition:

Definition 5.11. Let V be a complex vector space. A **Hermitian inner product** on V is a function

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{C}$$

such that

- $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for any $u, v \in V$,
- $\langle \cdot, \cdot \rangle$ is $\mathbb{C}$-linear in the first entry,
- $\langle v, v \rangle \geq 0$ for all v and equality holds if and only if $v = 0$.

Definition 5.12. Let X be a real manifold and let $\pi : E \longrightarrow X$ be a complex vector bundle on X. A **Hermitian metric** h on E is defined by a Hermitian inner product

$$\langle \cdot, \cdot \rangle_x : E(x) \times E(x) \longrightarrow \mathbb{C}$$

at each fiber $E(x)$ such that for any open set $U \subset X$ and for any pair of sections on U, $s, t \in C^\infty(U, E)$, the function

$$\langle s(\cdot), t(\cdot) \rangle : U \longrightarrow \mathbb{C} \qquad x \mapsto \langle s(x), t(x) \rangle_x$$

is smooth. A complex vector bundle E equipped with a Hermitian metric h is called a **Hermitian vector bundle** (E, h).

We now want to describe a Hermitian metric locally around a point. Let $\pi : E \longrightarrow X$ and h be as in the definition above and let $x \in X$. Let $s_1, \ldots, s_r$ be a local frame for E in a neighbourhood U of x. The Hermitian metric with respect to such a frame is represented by the $(r \times r)$-matrix of smooth functions $H = (h_{i,j})$, given by

$$h_{i,j}(x) := \langle s_i(x), s_j(x) \rangle_x.$$

Thus, if $\sigma, \sigma' \in C^\infty(U, E)$ are sections which, with respect to the frame $s_1, \ldots, s_r$, are represented by $f = (f_1, \ldots, f_r)^T$ and $f' = (f_1', \ldots, f_r')^T$ respectively, then

$$\langle \sigma(x), \sigma'(x) \rangle_x = f^T H \overline{f}'.$$

Note that, since h is Hermitian, we have $H^T = \overline{H}$.

Proposition 5.13. *Every complex vector bundle $\pi : E \longrightarrow X$ admits a Hermitian metric h.*

Before proving the proposition, we recall the definition of a partition of unity:

Definition 5.14. Let M be a manifold and let $\mathcal{U} = \{U_\alpha\}$ be an open covering. A **partition of unity** with respect to $\mathcal{U}$ is a collection of smooth functions $f_\alpha : M \longrightarrow [0, 1]$ such that

1. $\mathrm{Supp}(f_\alpha) \subset U_\alpha$ for all α (in particular, $f_\alpha = 0$ outside U_α),
2. $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$, and
3. for all $x \in M$, there exists an open nieghbourhood W of x such that $\mathrm{Supp}(f_\alpha) \cap W \neq 0$ only for finitely many α.

Proposition 5.15. *Let M be a manifold with open cover $\mathcal{U} = \{U_\alpha\}$, then there exists a partition of unity with respect to $\mathcal{U}$.*

The proof of this proposition fits better into a first course on manifolds, and we do not give it here.

Proof of Proposition 5.13. Let $\{U_\alpha\}$ be a trivializing cover for E, and let h_α be an Hermitian metric on the restriction of E to U_α. Let f_α be a partition of unity with respect to the open cover $\{U_\alpha\}$. Then we may define

$$h = \sum f_\alpha h_\alpha.$$

It is clear that, for every $x \in X$, this defines a Hermitian inner product on $E(x)$. Moreover, if $\sigma, \sigma' \in C^\infty(U, E)$, then the function

$$x \mapsto \langle \sigma(x), \sigma'(x) \rangle_x = \sum f_\alpha \langle \sigma(x), \sigma'(x) \rangle_{\alpha,x}$$

is smooth. Thus, h is a Hermitian metric on E. □

Definition 5.16. Let $\pi\colon E \longrightarrow X$ be a Hermitian vector bundle of rank r. Then, for each $p, q \geq 0$, the Hermitian metric induces a bilinear map

$$\begin{aligned} C^\infty(X, E \otimes \Omega^p_{X,\mathbb{C}}) \times C^\infty(X, E \otimes \Omega^q_{X,\mathbb{C}}) &\longrightarrow C^\infty(X, \Omega^{p+q}_{X,\mathbb{C}}) \\ (\sigma, \tau) &\mapsto \langle \sigma, \tau \rangle \end{aligned}$$

which is locally defined as follows. Let $x \in X$ and let $s_1, \ldots, s_r$ be a local frame of E in an open set U of X containing x. If $\sigma \in C^\infty(X, E \otimes \Omega^p_{X,\mathbb{C}})$ and $\tau \in C^\infty(X, E \otimes \Omega^q_{X,\mathbb{C}})$, then locally we can write

$$\sigma = \sum_{i=1}^r \sigma_i \otimes s_i \qquad \text{and} \qquad \tau = \sum_{i=1}^r \tau_i \otimes s_i$$

where σ_i and τ_i are smooth p-forms and q-forms on U respectively. Let H be the matrix associated to the Hermitian metric h with respect to the frame $s_1, \ldots, s_r$. Then we define:

$$\langle \sigma, \tau \rangle := \sigma^T H \overline{\tau} := \sum_{i,j=1}^r h_{i,j}\ \sigma_i \wedge \overline{\tau_j}.$$

Note that $\langle \sigma, \tau \rangle$ is a smooth $(p+q)$-form on U.

Definition 5.17. Let E be a Hermitian vector bundle on a real manifold X and let ∇ be a connection on E. We say that ∇ is **Hermitian** (or **compatible** with the Hermitian metric on E) if the following Leibnitz rule holds:

$$d\langle \sigma, \tau \rangle = \langle \nabla\sigma, \tau \rangle + \langle \sigma, \nabla\tau \rangle$$

for all $\sigma \in C^\infty(X, E)$ and $\tau \in C^\infty(X, E)$.

Proposition 5.18. *Let (E, h) be a Hermitian vector bundle on X. Then for all $x \in X$ there exist a neighbourhood $U \subset X$ of x and a local frame $s_1, \ldots, s_r \in C^\infty(U, E\mid_U)$ such that*

$$\langle s_i, s_j \rangle = \delta_{ij} \quad \textit{for} \quad i, j \in \{1, \ldots, r\}.$$

Proof. This is proved using the Gram-Schmidt orthogonalisation procedure, and we leave the details as an exercise. □

Lemma 5.19. *Let (E, h) be a Hermitian vector bundle with connection ∇. Let $\psi = (s_1, \ldots, s_r)$ be a local orthonormal frame. Write $\nabla = \psi^{-1}(\mathrm{d} + A)\psi$. Then ∇ is Hermitian if and only if*

$$A^T = -\overline{A}.$$

Proof. Since $s_1, \ldots, s_r$ is an orthormal frame, the matrix H associated to the Hermitian metric with respect to the metric is the identity matrix. Let $\sigma = \sum_{i=1}^r f_i s_i$, $\tau = \sum_{i=1}^r g_i s_i$ be two local C^∞-sections of E. Then

$$\langle \sigma, \tau \rangle = \sum_{i=1}^r f_i \overline{g}_i = f^T \overline{g},$$

where $f = (f_1, ..., f_r)^T$, $g = (g_1, ..., g_r)^T$. It follows,

$$d\langle \sigma, \tau \rangle = (df^T)\overline{g} + f^T(d\overline{g}).$$

We also have

$$\begin{aligned}\langle \nabla\sigma, \tau \rangle &= (Af + df)^T \overline{g} \\ &= f^T A^T \overline{g} + (df^T)\overline{g}\end{aligned}$$

and

$$\begin{aligned}\langle \sigma, \nabla\tau \rangle &= f^T(\overline{Ag + dg}) \\ &= f^T \overline{A}\overline{g} + f^T d\overline{g}.\end{aligned}$$

Thus, the Leibniz rule implies

$$f^T(A^T + \overline{A})\overline{g} = 0$$

for all f, g. Thus, the claim follows. □

Corollary 5.20. *Let (E, h) be a Hermitian vector bundle with connection ∇. Let H and A be the matrices associated to a Hermitian metric and a connection respectively, with respect to the same local frame, then the connection is compatible with the metric if and only if*

$$dH = A^T H + H\overline{A}, \tag{5.21}$$

where $dH = (dh_{i,j})$ denotes the differential of $H = (h_{i,j})$.

This is proved analogously to Lemma 5.19 and we leave the details as an exercise.

Theorem 5.22. *Let X be a manifold and let E be a Hermitian vector bundle on X. Then X admits a compatible connection ∇.*

Proof. Let ∇ be a connection on E (which exists by Corollary 5.4). We now modify the connection ∇ to make it Hermitian.
For any two C^∞-sections σ, τ of E, define

$$F(\sigma, \tau) = d\langle \sigma, \tau \rangle - \langle \nabla\sigma, \tau \rangle - \langle \sigma, \nabla\tau \rangle \in C^\infty(X, \Omega^1_{X,\mathbb{C}}).$$

Note that $F(\sigma, \tau) = \overline{F(\tau, \sigma)}$, and $F(h\sigma, \tau) = hF(\sigma, \tau)$, for each $h \in C^\infty(X \to \mathbb{C})$. In a sense, the tensor F measures failure of ∇ to preserve the Hermitian metric. Define a new connection $\widetilde{\nabla}$ by declaring

$$\langle \widetilde{\nabla}\sigma, \tau \rangle = \langle \nabla\sigma, \tau \rangle + \frac{1}{2}F(\sigma, \tau), \quad \sigma, \tau \in C^\infty(X, E).$$

Note that since τ above is arbitrary, this equality determines the value $\widetilde{\nabla}\sigma$ uniquely. Check that $\widetilde{\nabla}$ is a connection, i.e. satisfies the Leibniz rule.
We claim that the connection $\widetilde{\nabla}$ is Hermitian. Indeed, we have

$$\langle \widetilde{\nabla}\tau, \sigma \rangle = \langle \nabla\tau, \sigma \rangle + \frac{1}{2}F(\tau, \sigma),$$

and by applying complex conjugation, we get

$$\langle \sigma, \widetilde{\nabla}\tau \rangle = \langle \sigma, \nabla\tau \rangle + \frac{1}{2}F(\sigma, \tau).$$

Therefore,

$$\begin{aligned} d\langle \sigma, \tau \rangle - \langle \widetilde{\nabla}\sigma, \tau \rangle - \langle \sigma, \widetilde{\nabla}\tau \rangle &= \\ = d\langle \sigma, \tau \rangle - \langle \nabla\sigma, \tau \rangle - \langle \sigma, \nabla\tau \rangle - \frac{1}{2}F(\sigma, \tau) - \frac{1}{2}F(\sigma, \tau) &= \\ = F(\sigma, \tau) - \frac{1}{2}F(\sigma, \tau) - \frac{1}{2}F(\sigma, \tau) = 0. \end{aligned}$$

□

Note that the connection ∇ from Theorem 5.22 is not uniquely determined. In the proof we began by choosing a connection, but could have chosen another one. (It is, of course, possible that the correction in the proof would lead to the same resulting connection, but one can check that is not the case.) If one imposes one additional condition on the condition, then one can make it uniquely determined. This leads to the notion of *Chern connection* in the next section.

5.3 Chern connection

Definition 5.23. Let X be a complex manifold and E be a complex vector bundle over X. If $\nabla\colon C^\infty(X, E) \to C^\infty\left(X, \Omega^1_{X,\mathbb{C}} \otimes E\right)$ is a connection on E, then, by composing with the projections

$$p_{1,0} : \Omega^1_{X,\mathbb{C}} \to \Omega^{1,0}_X \qquad \text{and} \qquad p_{0,1} : \Omega^1_{X,\mathbb{C}} \to \Omega^{0,1}_X$$

we can decompose

$$\nabla = \nabla^{1,0} + \nabla^{0,1}$$

where

$$\nabla^{1,0}\colon C^\infty(X, E) \to C^\infty\left(X, \Omega^{1,0}_X \otimes E\right)$$

and

$$\nabla^{0,1}\colon C^\infty(X, E) \to C^\infty\left(X, \Omega^{0,1}_X \otimes E\right).$$

Recall that if the vector bundle E is additionally holomorphic, then it has an operator $\overline{\partial}_E$ of the same type as $\nabla^{0,1}$ (Proposition 4.43), and we can ask for $\nabla^{0,1}$ to agree with $\overline{\partial}_E$.

Theorem 5.24. *Let X be a complex manifold and let (E, h) be a Hermitian holomorphic vector bundle. Then there is a unique connection*

$$\nabla_E\colon C^\infty(X, E) \to C^\infty\left(X, \Omega^1_{X,\mathbb{C}} \otimes E\right)$$

such that

$$\nabla^{0,1}_E = \overline{\partial}_E$$

and such that ∇_E is compatible with the metric. Locally, if $\nabla = \psi^{-1}(\mathrm{d} + A)\psi$ and $H = (h_{ij})$, we have that $A = \overline{H}^{-1}\partial\overline{H}$.

Definition 5.25. We call ∇_E the **Chern connection** of (E, h). Moreover its curvature $\Theta_E := \Theta_{\nabla_E}$ is called **Chern curvature** of (E, h).

Proof. As above, we will define the connection locally and then show that it is independent of any choice made and so it extends to X.

Local existence: Let $x \in X$ be a point and let $s_1, \ldots, s_r$ be a local holomorphic frame defined over an open set U of X containing x. Let $H = (h_{i,j})$ be the matrix defining the metric on U with respect to $s_1, \ldots, s_r$, i.e. $h_{i,j} = \langle s_i, s_j \rangle$. We denote by ∂H the $(r \times r)$-matrix of $(1, 0)$-forms on U defined by

$$\partial H = (\partial h_{i,j})$$

We then define

$$A := \overline{H}^{-1} \partial \overline{H}, \tag{$*$}$$

and we consider the connection ∇_E on $E|_U$ defined by A, so that if $\sigma = \sum f_i \sigma_i$ is a section of E on U, where $f_1, \ldots, f_r$ are smooth functions, and $f = (f_1, \ldots, f_r)^T$, then we have

$$\nabla_E(f) = Af + df.$$

Note that $A = (a_{i,j})$ is defined by $(1, 0)$-forms $a_{i,j}$. In particular, it follows that on U we have

$$\nabla_E^{0,1} = \overline{\partial}_E.$$

In order to check that the connection is compatible with the metric h on U, we need to check that A satisfies the equation from Corollary 5.20, i.e.

$$dH = A^T H + H\overline{A}.$$

We have

$$\begin{aligned} A^T H &= (\overline{H}^{-1} \partial \overline{H})^T \cdot H \\ &= \partial \overline{H}^T (\overline{H}^{-1})^T H \\ &= (\partial H) H^{-1} H = \partial H \end{aligned}$$

where the third equality follows from the fact that $\overline{H} = H^T$, since h is Hermitian. Similarly,

$$H\overline{A} = H(H^{-1}\overline{\partial} H) = \overline{\partial} H.$$

Since $dH = \partial H + \overline{\partial} H$, it follows that the equation from Corollary 5.20 holds and the connection is compatible with the metric.

Local uniqueness: We now show that such connection is unique on U. Let ∇ be a connection on E_U which is compatible with the metric and such that $\nabla^{0,1} = \overline{\partial}_E$. Let $B = (b_{i,j})$ be the matrix associated to ∇ respect to the frame $s_1, \ldots, s_r$. We may write $B = B^{(1,0)} + B^{(0,1)}$ where $B^{(1,0)}$ (resp. $B^{(0,1)}$) is the $(r \times r)$-matrix obtained by taking the $(1, 0)$ (resp. $(0, 1)$) components of $b_{i,j}$. If σ is a section of E, then on U, we have $\sigma = \sum_{i=1}^r f_i s_i$ where $f_1, \ldots, f_r$ are smooth functions on U and if $f = (f_1, \ldots, f_r)$, we have

$$\overline{\partial} f + B^{0,1} f = \nabla^{0,1}(f) = \overline{\partial} f.$$

It follows that $B^{(0,1)} = 0$, i.e. $B = B^{(1,0)}$. Since ∇ is compatible with the metric, Corollary 5.20 implies that

$$dH = B^T H + H\overline{B}.$$

It follows that

$$\partial H = B^T H, \quad \text{and} \quad \overline{\partial} H = H\overline{B}.$$

Thus,

$$B = \overline{H}^{-1} \partial \overline{H} = A.$$

It follows that $\nabla = \nabla_E$ on U, i.e. the connection is unique on U.

Global existence and uniqueness: The previous part also implies that if ∇'_E is a connection on a different open set U' of X, which is compatible with the Hermitian metric and such that $\nabla'^{(0,1)}_E = \overline{\partial}_E$, then, on the intersection $U \cap U'$, the connection ∇_E must coincide with the connection ∇'_E. Therefore, the connection ∇_E extends uniquely over X. This proves the theorem. □

Corollary 5.26. *Let (E, h) be a Hermitian holomorphic vector bundle of rank r on a complex manifold X. Let ∇_E be the Chern connection on (E, h) and Θ_E its curvature. Let A be the matrix representing ∇_E with respect to some local holomorphic frame $s_1, \ldots, s_r$.*
Then,

1. *A is of type $(1, 0)$ and $\partial A = -A \wedge A$.*
2. *Locally $\Theta_E = \overline{\partial} A$ and, in particular, Θ_E is of type $(1, 1)$.*
3. *$\overline{\partial}\Theta_E = 0$.*

Proof. Let H be the matrix representing the metric h with respect to the local holomorphic frame $s_1, \ldots, s_r$. We first prove (1). By the local formula from 5.24 we have that A is of type $(1, 0)$ and

$$A = \overline{H}^{-1}\partial\overline{H}.$$

We have

$$0 = \partial(\overline{H}\overline{H}^{-1}) = \overline{H}\partial(\overline{H}^{-1}) + (\partial\overline{H})\overline{H}^{-1}.$$

Therefore,

$$\partial(\overline{H}^{-1}) = -\overline{H}^{-1}(\partial\overline{H})\overline{H}^{-1}.$$

Thus, since $\partial^2\overline{H} = 0$, we have

$$\begin{aligned}
\partial A &= \partial(\overline{H}^{-1}\partial\overline{H}) \\
&= \partial(\overline{H}^{-1}) \wedge \partial\overline{H} \\
&= -(\overline{H}^{-1}(\partial\overline{H})\overline{H}^{-1}) \wedge \partial\overline{H} \\
&= -(\overline{H}^{-1}\partial\overline{H}) \wedge (\overline{H}^{-1}\partial\overline{H}) = -A \wedge A.
\end{aligned}$$

We now prove (2). Recall that, by definition, $\Theta_E = A \wedge A + dA$. Thus, by (1) we have

$$\Theta_E = A \wedge A + \partial A + \overline{\partial} A = \overline{\partial} A.$$

Finally, we have

$$\overline{\partial}\Theta_E = \overline{\partial}^2 A = 0$$

and also (3) follows. □

In a certain sense, there is a *converse to the existence of the Chern connection.* This is the following Proposition that we state without giving its proof:

Proposition 5.27 ([1]). *Let (E, h) be a Hermitian vector bundle over a complex manifold X. Let ∇ be a Hermitian connection on E such that $F_\nabla \in C^\infty(X, \Omega_X^{1,1} \otimes \operatorname{End} E)$. Then there exists a natural holomorphic structure on E such that ∇ is its Chern connection.*

The two constructions are inverses of each other, and the correspondence between Hermitian connections with curvature of type $(1, 1)$ and holomorphic structures is one-to-one, if one quotients out by the right notion of isomorphism on both sides, which we also do not discuss here.

5.4 Fubini-Study form on $\mathbb{CP}^n$

In the following we will compute one example of a Chern curvature, namely of a connection on $\mathbb{CP}^n$. It may seem as if that is a randomly chosen example, but it will later turn out that this defines a Kähler structure on $\mathbb{CP}^n$, and through this induces Kähler structures on submanifolds of $\mathbb{CP}^n$. So, for practical purposes, this is often the *only* example of a Kähler structure one needs to know. Because of this, the formulae we derive for the Fubini-Study form often appear in complex geometry.

Recall that the tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ over $\mathbb{P}^n$ from Proposition 4.8 defined as

$$\mathcal{O}(-1) := \{(x, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \in x\}$$

Let $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$ be the standard affine cover of $\mathbb{CP}^n$ with local trivialisations

$$\begin{aligned} \psi_i : \pi^{-1}(U_i) &\to U_i \times \mathbb{C} \\ (x, (q_0, \dots, q_n)) &\mapsto (x, q_i). \end{aligned}$$

Their transition maps are $g_{ij} = \psi_i \psi_j^{-1} = \frac{x_i}{x_j}$. Define the metric h on $\mathcal{O}(-1)$ via

$$\begin{aligned} h_x : \mathcal{O}(-1)_x \times \mathcal{O}(-1)_x &\to \mathbb{C} \\ (v, w) &\mapsto \langle v, w \rangle_{\mathbb{C}^{n+1}}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^{n+1}}$ denotes the standard Hermitian inner product on $\mathbb{C}^{n+1}$. The bundle $\mathcal{O}(-1)$ is a line bundle, so a local frame consists just of a single section. Define a local frame on U_i by $s(x) = \psi_i^{-1}(x, 1)$. In this trivialisation, the metric has the form

$$h_{11}(x) = \langle s(x), s(x) \rangle_{\mathbb{C}^{n+1}} = \left\langle \frac{x}{x_i}, \frac{x}{x_i} \right\rangle_{\mathbb{C}^{n+1}} = \sum_{j=0}^n \left| \frac{x_j}{x_i} \right|^2. \tag{$*$}$$

We compute the Chern connection of $\mathcal{O}(-1)$. Consider the chart

$$\begin{aligned} (z_0, \dots, z_{i-1}, \widehat{z_i}, z_{i+1}, \dots, z_n) : U_i &\mapsto \mathbb{C}^n \\ [x_0 : \dots : x_n] &\mapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right), \end{aligned}$$

where $\widehat{\cdot}$ stands for omitting an entry. Writing $\nabla = \psi^{-1}(\mathrm{d} + A)\psi_i$, we have by Theorem 5.24 that $A = \overline{H}^{-1} \partial \overline{H}$. Thus

$$\begin{aligned} A &= \frac{\partial H}{H} \\ &= \frac{\partial \left(1 + \sum_{j \neq i} |z_j|^2\right)}{1 + \sum_{j \neq i} |z_j|^2} \\ &= \frac{\sum_{j \neq i} \mathrm{d}z_j \overline{z_j}}{1 + \sum_{j \neq i} |z_j|^2}, \end{aligned}$$

where in the first step we used that $H(x) \in \mathbb{R}$ for $x \in U_i$, so $\overline{H} = H$; in the third step we used $\partial(|z|^2) =$

$\partial(z\overline{z}) = \overline{z}\,\mathrm{d}z$. Hence

$$\begin{aligned}
\psi F_\nabla \psi^{-1} &= \mathrm{d}A + A \wedge A \\
&= \mathrm{d}A \\
&= \overline{\partial} A \\
&= \sum_{j \neq i} \left(\overline{\partial} \frac{\overline{z_j}}{1 + \sum_{m \neq i} |z_m|^2} \right) \wedge \mathrm{d}\, z_j \\
&= \sum_{j \neq i} \frac{(\overline{\partial} \overline{z}_j) \cdot (1 + \sum_{k \neq i} |z_k|^2) - \overline{z_i} \cdot \overline{\partial} \left(1 + \sum_{k \neq i} |z_k|^2\right)}{\left(1 + \sum_{m \neq i} |z_m|^2\right)^2} \wedge \mathrm{d}\, z_j \\
&= -\frac{1}{\left(1 + \sum_{m \neq i} |z_m|^2\right)^2} \cdot \sum_{j,k \neq i} f_{jk}\, \mathrm{d}\, z_j \wedge \mathrm{d}\, \overline{z_k},
\end{aligned}$$

where $f_{jk} = \left(1 + \sum_{m \neq i} |z_m|^2\right) \delta_{jk} - \overline{z_j} z_k$. Here, we used the local formula for the curvature from Proposition 5.10 in the first step; we used the fact that $\mathcal{O}(-1)$ is a line bundle, so $\operatorname{End} \mathcal{O}(-1) = \underline{\mathbb{C}}$ and therefore $A \in C^\infty(U_i, \Omega^1_{U_i,\mathbb{C}})$ and we have that $\alpha \wedge \alpha = 0$ for any 1-form α in the second step; in the third step we used that F_∇ is of type $(1, 1)$ by Corollary 5.26; in the fifth step we used the quotient rule for the derivative of a fraction.

Definition 5.28. The *Fubini-Study form* on $\mathbb{CP}^n$ is

$$\omega_{FS} := -\frac{i}{2\pi} F_\nabla \in C^\infty(\mathbb{CP}^n, \Omega^2_{X,\mathbb{C}} \otimes \operatorname{End}(\mathcal{O}(-1))) = C^\infty(\mathbb{CP}^n, \Omega^2_{X,\mathbb{C}}),$$

where ∇ is the Chern connection on $\mathcal{O}(-1)$ endowed with the Hermitian metric given by restricting the standard inner product on $\mathbb{C}^{n+1}$. (This formula was missing the factor $-\frac{i}{2\pi}$ in an earlier version of the notes, which was a typo.) By the above calculation we also have the alternative formula

$$\omega_{FS} = -\frac{i}{2\pi} \overline{\partial} \partial \log H,$$

where H is given locally in formula $(*)$ above. Even though the definition of H depends on the choice of affine patch, the calculation above shows that $\overline{\partial}\partial \log H$ is a well-defined 2-form on $\mathbb{CP}^n$, namely the Chern curvature, which is a globally defined object.
Notice because of $\overline{\partial}\partial = -\partial\overline{\partial}$, we have from the above formula also that

$$\omega_{FS} = \frac{i}{2\pi} \partial \overline{\partial} \log H.$$

5.5 Chern classes

Chern classes are numerical invariants for vector bundles. It turns out that they can be defined for *topological vector bundles*, but in this course we will only give the "Chern-Weil definition", which requires smoothness. The idea is to take $F_\nabla \in C^\infty(X, \Omega^2_{X,\mathbb{C}} \otimes \operatorname{End} E)$ and apply some operations to the endomorphism part to define forms on the manifold.

Definition 5.29. Let V be a complex vector space. Then we write

$$S^k(V)^* := \{P : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{C} : P \text{ multi-linear and symmetric}\},$$

where *symmetric* means that $P(\ldots, v, \ldots, w, \ldots) = P(\ldots, w, \ldots, v, \ldots)$. For $P \in S^k(V)^*$ denote by

$$\begin{aligned}
\widetilde{P} : V &\to \mathbb{C} \\
B &\mapsto P(B, \ldots, B)
\end{aligned}$$

the *polarised form* of P.

Lemma 5.30. *The map*

$$\begin{aligned} S^k(V)^* &\to \{Q \in \mathbb{C}[B] : Q \text{ homogeneous}\} \\ P &\mapsto \widetilde{P}(B) \end{aligned}$$

is bijective.

The proof is left as an exercise. We will be interested in the case of $V = \mathfrak{gl}(r, \mathbb{C}) = \mathbb{C}^{r\times r}$.

Definition 5.31. A symmetric map $P \in S^k(\mathfrak{gl}(r, \mathbb{C}))^*$

$$P : \mathfrak{gl}(r, \mathbb{C}) \times \cdots \times \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$$

is called Ad-*invariant* if for all $C \in \mathrm{GL}(r, \mathbb{C})$ and $B_1, \ldots, B_k \in \mathfrak{gl}(r, \mathbb{C})$ we have

$$P(CB_1C^{-1}, \ldots, CB_kC^{-1}) = P(B_1, \ldots, B_k). \tag{$*$}$$

Equivalently: $\widetilde{P}(CBC^{-1}) = \widetilde{P}(B)$ for all $C \in \mathrm{GL}(r, \mathbb{C})$ and $B \in \mathfrak{gl}(r, \mathbb{C})$. We denote the space of Ad-invariant symmetric maps by $(S^k(\mathfrak{gl}(r, \mathbb{C}))^*)^{\mathrm{GL}(r,\mathbb{C})}$.

Lemma 5.32. *An element $P \in S^k(\mathfrak{gl}(r, \mathbb{C}))^*$ is* Ad-*invariant if and only if for all $B, B_1, \ldots, B_k \in \mathfrak{gl}(r, \mathbb{C})$ we have*

$$\sum_{j=1}^k P(B_1, \ldots, B_{j-1}, [B, B_j], B_{j+1}, \ldots, B_k) = 0.$$

Proof. "$\Rightarrow$" Let $C = e^{tB}$, then $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}$ of $(*)$ gives:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} P(CB_1C^{-1}, \ldots, CB_kC^{-1}) \\ &= \sum_{j=1}^k P(CB_1C^{-1}|_{t=0}, \ldots, \frac{\mathrm{d}}{\mathrm{d}t} CB_jC^{-1}|_{t=0}, \ldots, CB_kC^{-1}) \\ &= \sum_{j=1}^k P(B_1, \ldots, [B, B_j], \ldots, B_k), \end{aligned} \tag{$**$}$$

where in the second step we used the fact that the differential of a multi-linear map $L : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is given by

$$\mathrm{d}_{x_1,\ldots,x_k} L(v_1, \ldots, v_k) = \sum_{j=1}^k L(x_1, \ldots, x_{j-1}, v_j, x_{j+1}, \ldots, x_k)$$

together with the chain rule; in the third step we used the product rule $\frac{\mathrm{d}}{\mathrm{d}t} e^{tB} B_j e^{-tB}|_{t=0} = BB_j - B_jB = [B, B_j]$.

"$\Leftarrow$" Let $C \in \mathrm{GL}(r, \mathbb{C})$ and $B_1, \ldots, B_k \in \mathfrak{gl}(r, \mathbb{C})$. The matrix exponential $\exp : \mathfrak{gl}(r, \mathbb{C}) \to \mathrm{GL}(r, \mathbb{C})$ is surjective, which can be seen by checking that it is surjective onto Jordan blocks, and then using $\exp(gXg^{-1}) = g\exp(X)g^{-1}$. Therefore, $C = e^B$ for some $B \in \mathfrak{gl}(r, \mathbb{C})$.

Then one checks as in $(**)$ that the function $f(t) := P(e^{tB}B_1e^{-tB}, \ldots, e^{tB}B_ke^{-tB})$ has

$$f'(t) = \sum_{j=1}^k P(e^{tB}B_1e^{-tB}, \ldots, [B, e^{tB}B_je^{-tB}], \ldots, e^{tB}B_ke^{-tB})$$

which is equal to zero by assumption. Thus, f is constant, which proves the claim. $\square$

Proposition 5.33. *Let $(S^k(\mathfrak{gl}(r,\mathbb{C}))^*)^{\mathrm{GL}(r,\mathbb{C})}$. Then for any complex vector bundle $E \to M$ and partition $m = i_1 + \cdots + i_k$ and $x \in M$ there exists a naturally induced multi-linear map*

$$P : \left(\Omega^{i_1}_{M,\mathbb{C}} \otimes \operatorname{End} E\right)_x \times \cdots \times \left(\Omega^{i_k}_{M,\mathbb{C}} \otimes \operatorname{End} E\right)_x \to \left(\Omega^m_{M,\mathbb{C}}\right)_x$$

defined by $P(\alpha_1 \otimes t_1, \ldots, \alpha_k \otimes t_k) = (\alpha_1 \wedge \cdots \wedge \alpha_k) \cdot P(t_1, \ldots, t_k)$.

Proof. In a point $x \in M$ we have $E_x \cong \mathbb{C}^r$. Under this identification, we can view $(t_i)_x \in \operatorname{End}(E_x) \cong \mathfrak{gl}(r,\mathbb{C})$, so $P(t_1, \ldots, t_k)$ is defined. Because P is Ad-invariant, this definition is independent of the choice of identification $E_x \cong \mathbb{C}^r$. □

This pointwise definition induces a map of global sections:

$$P : C^\infty(M, \Omega^{i_1}_{M,\mathbb{C}} \otimes \operatorname{End} E) \times \ldots C^\infty(M, \Omega^{i_k}_{M,\mathbb{C}} \otimes \operatorname{End} E) \to C^\infty(M, \Omega^m_{M,\mathbb{C}}).$$

The map P is *graded symmetric*, for example $P(\alpha_1 \otimes t_1, \alpha_2 \otimes t_2) = (-1)^{i_1} P(\alpha_2 \otimes t_2, \alpha_1 \otimes t_1)$. (We can only write this equation is $i_1 = i_2$.) If $i_1 = \cdots = i_k = 2$, then this means the map P is symmetric, and by Lemma 5.30 we can recover the symmetric map P from its polarised form $\widetilde{P}$ in this case. Later we will want to apply such a map to F_∇.

Lemma 5.34. *For $\gamma_j \in C^\infty(M, \Omega^{i_j}_{M,\mathbb{C}} \otimes \operatorname{End} E)$ we have*

$$\mathrm{d}\,(P(\gamma_1, \ldots, \gamma_k)) = \sum_{j=1}^k (-1)^{\sum_{l=1}^{j-1} i_l} P(\gamma_1, \ldots, \nabla\gamma_j, \ldots, \gamma_k),$$

where d *denotes the exterior differential on forms.*

Proof. In the proof of Proposition 5.33 we defined the map induced by P on the trivial bundle and used Ad-invariance to show that it was well defined, i.e. independent of the choice of trivialisation. Thus, it suffices to prove this lemma for the trivial bundle $E = M \times \mathbb{C}^r$. By linearity, it suffices to prove the claim for $\gamma_j = \alpha_j \otimes t_j$ where $\alpha_j \in C^\infty(M, \Omega^{i_j}_{M,\mathbb{C}})$ and $t_j \in C^\infty(M, \underline{\mathbb{C}^{r\times r}})$. We can even assume the t_j are constant, i.e. $\mathrm{d}\,t_j = 0$, every element in $C^\infty(M, \Omega^{i_j}_{M,\mathbb{C}} \otimes \operatorname{End} E)$ is still a sum of elements of the form $\alpha \otimes t$.
Then

$$\begin{aligned}\mathrm{d}\,(P(\gamma_1, \ldots, \gamma_k)) &= \mathrm{d}\,(\alpha_1 \wedge \cdots \wedge \alpha_k \cdot P(t_1, \ldots, t_k))\\ &= \sum_{j=1}^k (-1)^{\sum_{l=1}^{j-1} i_l} \alpha_1 \wedge \cdots \wedge \mathrm{d}\alpha_j \wedge \cdots \wedge \alpha_k \cdot P(t_1, \ldots, t_k)\\ &= \sum_{j=1}^k (-1)^{\sum_{l=1}^{j-1} i_l} P(\alpha_1 \otimes t_1, \ldots, (\mathrm{d}\alpha_j) \otimes t_j, \ldots, \alpha_k \otimes t_k),\end{aligned}$$

where in the first step we used the definition of P; in the second step we used the Leibniz rule for the exterior derivative together d together with the fact that $P(t_1, \ldots, t_k) \in C^\infty(M, \underline{\mathbb{C}})$ is constant because of the assumption that the t_j are constant.
Now, the induced connection ∇ on $\operatorname{End} E$ acts as $\nabla\gamma = \mathrm{d}\gamma + [A, \gamma]$. The claim then follows from plugging in $\mathrm{d} = \nabla - A$ together with Lemma 5.32, which implies that

$$\sum_{j=1}^k (-1)^{\sum_{l=1}^{j-1} i_l} P(\alpha_1 \otimes t_1, \ldots, [A, \alpha_j \otimes t_j], \ldots, \alpha_k \otimes t_k) = 0.$$ □

Corollary 5.35. *Let ∇ be a connection on a complex vector bundle $E \to M$ of rank r. Then for $P \in (S^k(\mathfrak{gl}(r,\mathbb{C}))^*)^{\mathrm{GL}(r,\mathbb{C})}$ we have that $\widetilde{P}(F_\nabla) \in C^\infty(M, \Omega^{2k}_{M,\mathbb{C}})$ is closed.*

Proof. This follows from the previous lemma together with the Bianchi identity $\nabla F_\nabla = 0$. $\square$

Thus, for $P \in (S^k(\mathfrak{gl}(r,\mathbb{C}))^*)^{\mathrm{GL}(r,\mathbb{C})}$ and a vector bundle E of rank r we have that $[\widetilde{P}(F_\nabla)] \in H^{2k}(M,\mathbb{C})$ is a well defined cohomology class. The following lemma proves that this class is independent of the choice of connection:

Lemma 5.36. *If ∇ and ∇' are two connections on a complex vector bundle $E \to M$, then*

$$[\widetilde{P}(F_\nabla)] = [\widetilde{P}(F_{\nabla'})] \in H^{2k}(M,\mathbb{C}).$$

Proof. Let $\nabla' = \nabla + a$ for some $a \in C^\infty(M, \Omega^1_{M,\mathbb{C}} \otimes \operatorname{End} E)$.
We first show that

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \left[\widetilde{P}(F_{\nabla+ta})\right] = 0, \tag{$*$}$$

i.e. $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\widetilde{P}(F_{\nabla+ta})$ is an exact form.
We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}F_{\nabla+ta} &= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(F_\nabla + t\nabla a + t^2 a\wedge a) = \nabla a, \qquad \text{and therefore}\\ \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\widetilde{P}(F_{\nabla+ta}) &= (\mathrm{d}\widetilde{P})_{F_\nabla}(\nabla a) = k\cdot P(F_\nabla,\dots,F_\nabla,\nabla a),\end{aligned}$$

where in the first step we used the chain rule and in the second step we used the explicit description of the differential of a k-linear map. The right hand side can be identified as

$$\begin{aligned}k\cdot P(F_\nabla,\dots,F_\nabla,\nabla a) &= k\cdot(\mathrm{d}P(F_\nabla,\dots,F_\nabla,a) - P(\nabla F_\nabla,\dots,F_\nabla,A) - \dots - P(F_\nabla,\dots,\nabla F_\nabla,a))\\ &= k\cdot \mathrm{d}P(F_\nabla,\dots,F_\nabla,a),\end{aligned}$$

which is exact. This proves $(*)$.
We can now conclude the proof as follows:

$$\begin{aligned}\widetilde{P}(F_{\nabla+a}) &= \widetilde{P}(F_\nabla) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}|_{t=s}\widetilde{P}(F_{\nabla+ta)})\,\mathrm{d}s\\ &= \widetilde{P}(F_\nabla) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\widetilde{P}(F_{(\nabla+sa)+ta)})\,\mathrm{d}s\\ &= \widetilde{P}(F_\nabla) + \int_0^1 \mathrm{d}(\eta_s)\,\mathrm{d}s \qquad \text{for some } \eta_s \in C^\infty(M,\Omega^{2k-1}_{M,\mathbb{C}})\\ &= \widetilde{P}(F_\nabla) + \mathrm{d}\left(\int_0^1 \eta_s\,\mathrm{d}s\right),\end{aligned}$$

where in the first step we used the fundamental theorem of calculus; in the third step we used $(*)$, now with starting point $\nabla + sa$ instead of ∇; and in the last step we used that the exterior derivative and integration commute (this is a consequence of the Leibniz integral rule). $\square$

Definition 5.37. The above construction defines a homomorphism

$$(S^k(\mathfrak{gl}(r,\mathbb{C}))^*)^{\mathrm{GL}(r,\mathbb{C})} \to H^{2k}(M,\mathbb{C})$$

for a vector bundle $E \to M$ of rank r. We can extend this to an algebra homomorphism

$$\mathcal{W}_E : (S^*(\mathfrak{gl}(r,\mathbb{C}))^*)^{\mathrm{GL}(r,\mathbb{C})} \to H^{2*}(M,\mathbb{C}),$$

which is called *Weil homomorphism.*

Definition 5.38 (Chern classes). Let $\{\widetilde{P}_k\}$ be the homogeneous polynomial of degree k defined by

$$\det(\operatorname{Id} + B) = 1 + \widetilde{P}_1(B) + \cdots + \widetilde{P}_r(B).$$

The $\widetilde{P}_k$ are Ad-invariant. The *Chern forms* of a vector bundle $E \longrightarrow M$ of rank r with connection ∇ are

$$c_k(E, \nabla) := \widetilde{P}_k\left(\frac{i}{2\pi} F_\nabla\right) \in C^\infty(M, \Omega^{2k}_{M,\mathbb{C}}).$$

The *k-th Chern class* of the vector bundle E is the induced cohomology class

$$c_k(E) := [c_k(E, \nabla)] \in H^{2k}(M, \mathbb{C}).$$

The *total Chern class of* E is $c(E) := c_0(E) + c_1(E) + c_2(E) + \cdots \in H^{2*}(M, \mathbb{C})$.

Note that $c_0(E) = 1$ and $c_k(E) = 0$ for $k > \operatorname{rank} E$. We now want to derive an explicit formula for the Chern classes $c_k(E)$. To this end, we need an explicit formula for $\widetilde{P}_k$.
Let $B \in \mathfrak{gl}(r, \mathbb{C})$ have eigenvalues $\lambda_1, \ldots, \lambda_r$ (with repetition), then

$$\det(\operatorname{Id} + B) = \prod_{j=1}^{r} (1 + \lambda_j).$$

Let $\sigma_k(\lambda_1, \ldots, \lambda_r)$ be the k-th elementary symmetric polynomial, i.e.

$$\begin{aligned} \sigma_0(\lambda_1, \ldots, \lambda_r) &:= 1, \\ \sigma_k(\lambda_1, \ldots, \lambda_r) &:= \text{sum of all products of } k \text{ distinct variables} \\ &= \sum_{1 \le i_1 < \cdots < i_k \le r} \lambda_{i_1} \ldots \lambda_{i_k}, \end{aligned}$$

that is

$$\begin{aligned} \sigma_1(\lambda_1, \ldots, \lambda_r &= \lambda_1 + \cdots + \lambda + r = \operatorname{tr}(B), \\ \sigma_2(\lambda_1, \ldots, \lambda_r) &= \sum_{1 \le i < j \le r} \lambda_i \lambda_j = \frac{1}{2}\left(\operatorname{tr}(B)^2 - \operatorname{tr}(B^2)\right) \end{aligned}$$

etcetera, where the formula for σ_2 is one of the *Newton identities*. (The Newton identities also give formula for all higher elementary symmetric polynomials.) Hence:

$$\begin{aligned} c_1(E) &= \left[\frac{i}{2\pi} \operatorname{tr}(F_\nabla)\right] \in C^\infty(M, \Omega^2_{M,\mathbb{C}}), \\ c_2(E) &= \left[\left(-\frac{1}{4\pi^2}\right) \sum_{1 \le i < j \le r} \operatorname{tr}(F_\nabla) \wedge \operatorname{tr}(F_\nabla) - \operatorname{tr}(F_\nabla \wedge F_\nabla)\right] \in C^\infty(M, \Omega^4_{M,\mathbb{C}}), \end{aligned}$$

where all traces are taking in the endomorphism bundle $\operatorname{End} E$.

Proposition 5.39 (Properties of Chern classes).

1. *If $E_1, E_2 \longrightarrow M$ are isomorphic vector bundles over M, then*

$$c(E_1) = c(E_2).$$

2. *(*Naturality*) Let $\phi : N \longrightarrow M$ be smooth, and $E \longrightarrow M$ be a complex vector bundle. Then*

$$c(\phi^* E) = \phi^*(c(E)).$$

3. *If $E_1, E_2 \longrightarrow M$ are complex vector bundles, then*

$$c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2),$$

where $\cdot$ denotes the product in de Rham cohomology, namely $[\omega]\cdot[\sigma] := [\omega\wedge\sigma]$ for $\omega, \sigma \in C^\infty(M, \Omega^*_{M,\mathbb{C}})$. For the individual Chern classes, that means:

$$c_k(E_1 \oplus E_2) = \sum_{j=0}^{k} c_j(E_1) \cdot c_{k-j}(E_2).$$

4. *Let $E \longrightarrow M$ be a complex vector bundle. Then*

$$c_k(E^*) = (-1)^k c_k(E).$$

5. *For the dual of the tautological bundle on $\mathbb{CP}^1$ we have*

$$c_1(\mathcal{O}(1)) = [\omega_{FS}] \in H^2(\mathbb{CP}^1, \mathbb{C}).$$

Proof.

1. If ∇ is a connection on E_1, and $\Phi : E_1 \longrightarrow E_2$ is a bundle isomorphism, then $\Phi \circ \nabla \circ \Phi^{-1}$ is a connection on E_2. If ψ is a local trivialisation of E_1, then $\psi \circ \Phi^{-1}$ is a local trivialisation of E_2. In these trivialisations, the connections ∇ and $\Phi \circ \nabla \circ \Phi^{-1}$ have the same local formula. The definition of the Weil homomorphism was local, so $W_E(\widetilde{P}_k) = W_F(\widetilde{P}_k)$.

2. Given a connection ∇ on E, we have a natural connection $\phi^*\nabla$ on ϕ^*E, whose curvature satisfies

$$\phi^*(F_\nabla) = F_{\phi^*\nabla}.$$

(Exercise.) Then

$$c_k(\phi^*E) = \widetilde{P}_k\left(\frac{i}{2\pi} \cdot F_{\phi^*\nabla}\right) = \widetilde{P}_k\left(\frac{i}{2\pi} \cdot \phi^*(F_\nabla)\right) = \phi^*\widetilde{P}_k\left(\frac{i}{2\pi} F_\nabla\right) = \phi^* c_k(E).$$

3. The $\widetilde{P}_k$ were defined as $\det(\mathrm{Id} + B) = 1 + \widetilde{P}_1(B) + \cdots + \widetilde{P}_r(B)$. We have for $A \in \mathbb{C}^{r_1\times r_1}$ and $B \in \mathbb{C}^{r_2\times r_2}$:

$$\begin{aligned} 1 + \widetilde{P}_1\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \cdots + \widetilde{P}_r\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} &= \det\left(\mathrm{Id} + \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) \\ &= \det(\mathrm{Id} + A)\det(\mathrm{Id} + B) \\ &= (1 + \widetilde{P}_1(A) + \cdots + \widetilde{P}_r(A))(1 + \widetilde{P}_1(B) + \cdots + \widetilde{P}_r(B)), \end{aligned}$$

so

$$\widetilde{P}_k\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sum_{j=0}^{k} \widetilde{P}_j(A)\widetilde{P}_{k-j}(B). \tag{$*$}$$

Now let ∇^1, ∇^2 be connections on E_1 and E_2 respectively. Then $\nabla^1 + \nabla^2$ is a connection on $E_1 \oplus E_2$ with

$$F_{\nabla^1+\nabla^2} = \begin{pmatrix} F_{\nabla^1} & 0 \\ 0 & F_{\nabla^2} \end{pmatrix}.$$

Therefore, by $(*)$:

$$c_k(E \oplus F) = \widetilde{P}_k(F_{\nabla_1+\nabla_2}) = \sum_{j=1}^{k} \widetilde{P}_j(F_{\nabla^1})\widetilde{P}_{k-j}(F_{\nabla^2}) = \sum_{j=0}^{k} c_j(E_1) \cdot c_{k-j}(E_2).$$

4. The bundle E^* admits a natural connection ∇^* satisfying $F_{\nabla^*} = -F_\nabla^T$ (exercise). We have

$$\widetilde{P}_k(-A) = (-1)^k \widetilde{P}_k(A)$$

by multilinearity of P_k and

$$\widetilde{P}_k(A^T) = \widetilde{P}_k(A)$$

by transpose invariance of det. This implies

$$\widetilde{P}_k(F_{\nabla^*}) = \widetilde{P}_k(-F_\nabla^T) = (-1)^k \widetilde{P}_k(F_\nabla).$$

5. By definition of ω_{FS} from Definition 5.28 we have $\omega_{\mathrm{FS}} = F_\nabla$, where ∇ was the Chern connection (with respect to some Hermitian metric) on $\mathcal{O}(1)$. This proves the claim. □

It is an observation by Grothendieck, that these conditions uniquely define Chern classes. Hence, an alternative definition of Chern classes is the following: let c_k be maps satisfying

1. $c_k(E) \in H^{2k}(M, \mathbb{C})$ with $c_0(E) = 1$ and $c_k(E) = 0$ for $k > \operatorname{rank} E$;
2. for $\phi : N \to M$ smooth and a complex vector bundle $E \to M$ we have $\phi^*(c(E)) = c(\phi^*(E))$;
3. for complex vector bundles $E_1, E_2 \to M$ we have $c(E_1 \oplus E_2) = c(E_1)c(E_2)$;
4. $c_1(\mathcal{O}(1)) = [\omega_{\mathrm{FS}}]$.

Here, one can even replace smooth vector bundles by continuous vector bundles, and the de Rham cohomology by the singular cohomology with integer coefficients.
We briefly mention one application of Chern classes, namely *stable bundles*.

Definition 5.40. Let $E \to \mathbb{CP}^n$ be a holomorphic vector bundle. Then

$$\deg E := \int_{\mathbb{CP}^n} c_1(E) \wedge \omega_{\mathrm{FS}}^{n-1}$$

is called the *degree of E* and

$$\mu(E) := \frac{\deg E}{\operatorname{rank} E}$$

is called the *slope of E*. The bundle E is called *stable*, if for every proper holomorphic subsheaf $F \subset E$ we have $\mu(F) < \mu(E)$.

Note that we have not defined sheaves in this lecture. Sometimes it is enough to think of *subbundles* $F \subset E$ instead of *subsheaves*, but in general one needs sheaf theory.
This definition can be generalised to algebraic varieties in $\mathbb{CP}^n$, not just $\mathbb{CP}^n$ itself. The set of all stable bundles is an interesting invariant of a complex manifold. It appears in homological mirror symmetry, which is a conjecture from physics. *Counting* stable bundles in a precise sense is the object of Donaldson-Thomas theory.

Example 5.41. Line bundles are stable.

Definition 5.42. Let $E \to \mathbb{CP}^n$ be a Hermitian vector bundle and ∇ be a Hermitian connection on E. Then ∇ is called *Hermite-Einstein* connection if $F_\nabla^{0,2} = 0$ and $F_\nabla \wedge \omega_{\mathrm{FS}}^{n-1} = \lambda \cdot \omega_{\mathrm{FS}}^n$ for some $\lambda \in \mathbb{C}$.

Example 5.43. On the bundle $\mathcal{O}(1) \to \mathbb{CP}^n$, the Chern connection ∇ satisfies $F_\nabla^{0,2} = 0$ and $F_\nabla \wedge \omega_{\mathrm{FS}}^{n-1} = 1 \cdot \omega_{\mathrm{FS}}^n$, so ∇ is Hermite-Einstein. More is true (though that is not obvious): every line bundle admits a Hermite-Einstein connection.

This can be generalised to *Kähler manifolds*, which will be introduced in the next section. Hermite-Einstein connections in complex dimension two are essentially the same as anti-self-dual connections, which are studied in Donaldson theory. The last two examples say all line bundles are stable and they always admit Hermite-Einstein connections. This is a special case of the following general theorem:

Theorem 5.44 (Donaldson). *Let $E \to \mathbb{CP}^n$ be a holomorphic bundle, then E is stable if and only if it admits a Hermite-Einstein connection.*

This theorem is known under the names *Donaldson-Uhlenbeck-Yau theorem* and *Kobayashi-Hitchin correspondence*. Like the definitions before, this theorem also holds more generally for Kähler manifolds.
Another application of characteristic classes are so-called *index theorems*. Here is the simplest example of an index theorem:

Theorem 5.45 (Riemann-Roch formula). *Let X be a complex manifold of complex dimension one and compact. Assume the underlying real surface has genus $g \geq 0$. Let $E \to X$ be a holomorphic rank r vector bundle. Then*

$$h^0(X, E) - h^1(X, E) = \int_X c_1(E) + r(1 - g).$$

This theorem is surprising and useful for the following reason: roughly speaking, the left hand side counts solutions to a partial differential equation. The number $h^0(X, E)$ is the dimension of the space of solutions for the equation $\overline{\partial}_E s = 0$. (The *correction term* $h^1(X, E)$ is less interesting, but it is equal to zero in many cases.) This quantity is very hard to compute. On the other hand, the right hand side does not depend on the complex structure of the manifold, but is only an integral over a smooth manifold, which can often be easily computed.
This theorem is a special case of the *Atiyah-Singer index theorem* for elliptic differential operators on any manifold, not necessarily complex. In this index theorem, other more complicated characteristic classes appear.

6 Kähler manifolds

Definition 6.1. Let $V \subset \mathbb{C}^n$ be open. A form $\omega \in C^\infty(V, \Omega_V^{1,1})$ is called *real* if $\overline{\omega} = \omega$. It is called *positive*, if in the representation

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk} \, \mathrm{d}z_j \wedge \mathrm{d}\overline{z_k}$$

the matrix-valued function $(h_{jk}) : V \to \mathbb{C}^{n \times n}$ is positive definite in each point. Equivalently, $\omega_x(v, i \cdot v) > 0$ for all $v \in \mathbb{C}^n$ for all $x \in V$.

Definition 6.2. Let X be a complex manifold of dimension n and $\omega \in C^\infty(X, \Omega_X^{1,1})$. Then ω is called *positive* if for all $x \in X$ there exists a holomorphic chart (U, ϕ), $\phi : U \to V \subset \mathbb{C}^n$ such that $(\phi^{-1})^* \omega$ is a positive form on V.

Definition 6.3. Let X be a complex manifold and $\omega \in C^\infty(X, \Omega_X^{1,1})$ be positive. Then ω is *Kähler* if $\mathrm{d}\omega = 0$. We call the pair (X, ω) a *Kähler manifold*.

Example 6.4. On $X = \mathbb{C}^n$ we have that

$$\omega = \frac{i}{2} \sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\overline{z_j}$$

is a real, positive $(1, 1)$-form with $\mathrm{d}\omega = 0$. Hence $(\mathbb{C}^n, \omega)$ is Kähler.

Example 6.5. Let $\Lambda \subset \mathbb{C}^n$ be a lattice and $X = \mathbb{C}^n / \Lambda$ the complex torus. Then ω from Example 6.4 descends to X, i.e. for $\lambda \in \Lambda$ and

$$\begin{aligned} \phi_\lambda : \mathbb{C}^n &\to \mathbb{C}^n \\ x &\mapsto x + \lambda \end{aligned}$$

we have $\phi^* \omega = \omega$. Hence, X with this 2-form is also a Kähler manifold.

Example 6.6. On $X = \mathbb{CP}^n$ we have that ω_{FS} is Kähler. (Exercise.)

Example 6.7. Every complex manifold of dimension one admits a Kähler form. (Exercise.)

Proposition 6.8. *Let (X, ω) be Kähler and $Y \subset X$ be a complex submanifold. Then $(Y, \omega\mid_Y)$ is Kähler. In particular: complex submanifolds of $\mathbb{CP}^n$ are Kähler.*

Proof. Writing the inclusion as $i : Y \to X$, we have

$$\mathrm{d}(\omega\mid_Y) = \mathrm{d}(i^*\omega) = i^*(\mathrm{d}\omega) = 0.$$

Furthermore, ω is positive if and only if $\omega(v, Jv) > 0$ for all $v \in TX$, where J denotes the almost complex structure induced by the complex structure on X. In particular this holds for $v \in TY \subset TX$, so $\omega|_Y$ is still positive, which proves the claim. $\square$

We now include a brief reminder about integration and Stokes' theorem.
On an $\mathbb{R}$-vector space V, define

$$\mathcal{B} = \{(v_1, \ldots, v_n) \text{ basis of } V\}.$$

Define an equivalence relation $\sim$ on $\mathcal{B}$ as follows: for two bases $b, b' \in \mathcal{B}$, set

$$b \sim b' \quad \text{if and only if} \quad Ab = b' \text{ for some } A \in \mathrm{GL}(V) \text{ with } \det A > 0.$$

An equivalence class $[b]$ for some $b \in \mathcal{B}$ is called orientation of V.
A collection $\{O_{T_yM}\}_{y \in M}$ is called an *orientation of* M, if for all $y \in M$ the element O_{T_yM} is an orientation of T_yM, together with an atlas of M in which

$$\left(\frac{\partial}{\partial x_1}(y), \ldots, \frac{\partial}{\partial x_n}(y)\right) \in O_{T_yM} \text{ for every chart and for all } y \in M.$$

Now let M be oriented and $\omega \in \mathbb{C}^\infty(M, \Omega^n)$ with $\operatorname{supp}(\omega) \subset U$ for some positively oriented chart (U, φ). Then

$$\int_u \omega := \int_{\varphi(U)} (\omega_\varphi \circ \varphi^{-1})\, \mathrm{d}\lambda^n, \tag{$*$}$$

where $\omega_\varphi \in C^\infty(U)$ is given by $\omega = \omega_\varphi\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, i.e.

$$\omega_\varphi(x) = \omega_x\left(\frac{\partial}{\partial x_1}(x), \ldots, \frac{\partial}{\partial x_n}(x)\right).$$

Proposition 6.9. *The expression $\int_U \omega$ from $(*)$ is well defined, i.e. does not depend on the choice of (U, φ).*

The proof is an application of the transformation formula for the Lebesgue integral and is omitted here. In general, $\operatorname{supp}\omega$ is *not* contained in a chart. The define the general case, let $\{(U_\alpha, \varphi_\alpha)\}$ be a positively oriented atlas and f_α be a partition of unity for it. Then define

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} f_\alpha \cdot \omega. \tag{$**$}$$

Proposition 6.10. *The expression $\int_M \omega$ from $(**)$ is well defined, i.e. does not depend on the choice partition of unity.*

The proof uses the linearity of the Lebesgue integral and we also omit it here. Analogously one defines integration on manifolds with boundary, but we omit this due to long notation.

Definition 6.11. Let M be a manifold with orientation

$$O_M = \{O_{T_yM} : O_{T_yM} \text{ orientation of } T_yM\}$$

and boundary ∂M. Let $x \in \partial M$, $w = [\gamma] \in T_xM$, i.e. $\gamma : (-\epsilon, 0] \to M$ with $\gamma(0) = x$ and $\gamma'(0) \neq 0$. Then

$$O_{\partial M} := \{O_{T_x\partial M} := [(v_1, \ldots, v_{n-1}] : (w, v_1, \ldots, v_{n-1}) \in O_{T_xM}\}$$

is called the *induced orientation on* ∂M.

Theorem 6.12 (Stokes' theorem). *Let M be an oriented manifold with boundary ∂M and its induced orientation and $\omega \in C^\infty(M, \Omega^{n-1})$ with compact support. Then*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

This proof uses Fubini's theorem and integration by parts on the interval, and we also omit it here. We can now use integration to prove the following property about Kähler manifolds:

Lemma 6.13. *Let (X, ω) be a compact Kähler manifold of dimension n. Then $b_{2k} := \dim_{\mathbb{C}} H^{2k}(X, \mathbb{C}) > 0$ for $k \in \{1, \ldots, n\}$.*

Proof. We have $\mathrm{d}(\omega^k) = 0$ by the Leibniz rule, so it remains to check that $[\omega^k] \neq 0 \in H^{2k}(X, \mathbb{C})$.
Let $x \in X$ and (U, φ) be a complex chart around x, then

$$\omega(x) = \sum_{j,k=1}^{n} h_{jk}\, \mathrm{d}z_j(x) \wedge \mathrm{d}\overline{z_k}(x)$$

for some positive definite h_{jk}. The basis

$$\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \overline{z_1}}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z_n}}\right)$$

is positively oriented in the orientation from Proposition 4.25. We have

$$\omega^n(x) = \det(h_{jk})\, \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \ldots \mathrm{d}z_n \wedge \mathrm{d}\overline{z_n}$$

by the Leibniz formula for the determinant. Because h_{jk} is positive definite, we have that $\det(h_{jk}) > 0$, hence ω^n is nowhere zero and

$$\int_X \omega^n = \sum_\alpha \int_{\varphi_\alpha(U_\alpha)} \det(h_{jk}) \cdot f_\alpha > 0,$$

where f_α is a partition of unity for the complex atlas $(U_\alpha, \varphi_\alpha)$. Here, we used the definition of integral together with the fact that φ_α is positively oriented in the first step.
Now assume that $\omega^k = \mathrm{d}\eta$ for $\eta \in C^\infty(X, \Omega^{2k-1}_{X,\mathbb{C}})$. Then

$$0 < \int_X \omega^n = \int_X \mathrm{d}(\eta \wedge \omega^{n-k}) = \int_{\partial X} \eta \wedge \omega^{n-k} = 0,$$

where we used Stokes' theorem in the third step. This is a contraction, which proves the claim. □

Example 6.14 (Hopf surface). Fix $\lambda \in \mathbb{R}$, $0 < \lambda < 1$. Let $X = (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$, where the group action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$ is defined as

$$k \cdot z := \lambda^k z.$$

The following diffeomorphism is called *polar coordinates*:

$$\begin{aligned} S^3 \times \mathbb{R}_{>0} &\to \mathbb{C}^2 \setminus \{0\} \\ (x, r) &\mapsto r \cdot x. \end{aligned}$$

In these coordinates, the group action of $\mathbb{Z}$ can be written as:

$$k \cdot (x, r) = (x, \lambda^k r) \quad \text{for} \quad k \in \mathbb{Z}, (x, r) \in S^3 \times \mathbb{R}_{>0}. \tag{$*$}$$

Hence,

$$(S^3 \times \mathbb{R}_{>0})/\mathbb{Z} \cong S^3 \times (\mathbb{R}_{>0}/\mathbb{Z}) \cong S^3 \times S^1,$$

where $\cong$ denotes diffeomorphism, and we used $(*)$ in the first step. Now

$$\begin{aligned} b^2(X) &= b^2(S^3 \times S^1) \\ &= b^0(S^3)\,\underbrace{b^2(S^1)}_{=0} + \underbrace{b^1(S^3)}_{=0}\,b^1(S^1) + \underbrace{b^2(S^3)}_{=0}\,b^0(S^1) = 0, \end{aligned}$$

where we used the following facts from algebraic topology: in the second step, we used the *Künneth formula*; in the third step we used $b^1(S^3) = b^2(S^3) = 0$. (The property $b^2(S^1) = 0$ is a standard property of the de Rham cohomology.) Thus, by Lemma 6.13, the manifold X is not Kähler.
On the other hand, $\mathbb{Z}$ acts through biholomorphisms on $\mathbb{C}^2 \setminus \{0\}$, so X is a complex manifold.
Hence, X is an example of a complex manifold that is not Kähler.

6.1 The Hodge $\star$ operator

Definition 6.15. Let W be an $\mathbb{R}$-vector space with real inner product $\langle \cdot, \cdot \rangle : W \times W \to \mathbb{R}$. This induces an inner product on $\bigwedge^k W$ defined by

$$\langle v_1 \wedge \cdots \wedge v_k, v_1' \wedge \cdots \wedge v_k' \rangle := \det(\langle v_i, v_j' \rangle).$$

If $\dim_{\mathbb{R}}(W) = m$, then up to multiplication by (-1) there exists a unique $\omega \in \bigwedge^m W$ such that $\langle \omega, \omega \rangle = 1$. Then, for each $k \geq 0$ there exists $\star : \bigwedge^k W \to \bigwedge^{m-k} W$ such that

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega \quad \text{for} \quad \alpha, \beta \in \bigwedge^k W.$$

If $e_1, \ldots, e_m$ is an orthonormal basis of W such that $e_1 \wedge \cdots \wedge e_m = \omega$, then

1. $\star 1 = \omega$,
2. $\star e_1 = e_2 \wedge \cdots \wedge e_m$,
3. $\star\omega = 1$,
4. $\star e_i = (-1)^{i-1} e_1 \wedge \ldots e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_m$,
5. if $I \subset \{1, \ldots, m\}$ and I^c is the complement of U, then

$$\star e_I = \epsilon(\sigma) \cdot e_{I^c}, \tag{$*$}$$

   where $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $I = (i_1, \ldots, i_k)$ and $\epsilon(\sigma)$ is the sign of the permutation sending $(1, \ldots, m)$ to (I, I^c).

Proposition 6.16. *Let (X, ω) be Kähler and write $J : TX \to TX$ for the almost complex structure induced by the complex structure of X. Then for all $x \in X$ we have that*

$$\begin{aligned} g_x : T_xX \times T_xX &\to \mathbb{R} \\ (u, v) &\mapsto \omega(u, Jv) \end{aligned}$$

is a symmetric, positive definite bilinear form and $h_x(u, v) := g_x(u, v) + i\omega_x(u, v)$ is a Hermitian inner product.

Proof. **Symmetry**: For local complex coordinates $(z_1, \ldots, z_n)$ write $z_i = x_i + iy_i$. Then one checks that by explicit calculation that

$$\omega\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right) = \omega\left(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}\right),$$
$$\omega\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_l}\right) = \omega\left(\frac{\partial}{\partial y_k}, -\frac{\partial}{\partial x_l}\right),$$

which by linearity implies $\omega(Jv, Ju) = \omega(v, u)$ for all $u, v \in TX$. Hence

$$g(u, v) = \omega(u, Jv) = -\omega(Jv, u) = -\omega(J^2v, Ju) = \omega(v, Ju) = g(v, u).$$

Real: $\overline{g(u, v)} = \overline{\omega(u, Jv)} = \omega(u, Jv) = g(u, v)$ because ω is real.
Positive definite: $g(u, u) = \omega(u, Ju) > 0$ for $u \neq 0$ because ω is positive definite.
The claim for h follows from the properties of g and ω. □

Remark 6.17 (Connection to Riemannian geometry). The object g is called *Riemannian metric*. Given a Riemannian metric, there exists a canonical real connection ∇ on the $\mathbb{R}$-vector bundle $E = TX$. This induces a connection ∇ on $C^\infty(X, \Omega^k_{X,\mathbb{C}})$ that satisfies $\nabla\omega = 0$. (This is an equivalent definition of Kähler manifolds.)

Definition 6.18. Let (X, ω) be Kähler. Then $(T_{\mathbb{C}}X, h)$ is a Hermitian vector bundle, where h was defined in Proposition 6.16. This induces metrics on $T^*_{\mathbb{C}}X$ and $\Omega^k_{X,\mathbb{C}}$ by the same formula as in Definition 6.15. An explicit formula for this inner product is:

$$\langle \sigma_x, \tau_x \rangle_{h(x)} = \sum_{1 \leq i_1 < \cdots < i_k \leq 2n} \sigma_x(e_{i_1}, \ldots, e_{i_k}) \cdot \tau_x(e_{i_1}, \ldots, e_{i_k}),$$

where $e_1, \ldots, e_{2n}$ is an orthonormal basis of T_xX with respect to g_x.
Define $\mathrm{vol} := C \cdot \omega^n \in C^\infty(X, \Omega^{2n}_{X,\mathbb{C}})$, where $C \in \mathbb{R}$ is chosen so that $\langle \mathrm{vol}_x, \mathrm{vol}_x \rangle_x = 1$ for all $x \in X$. (Exercise: check that $\langle \omega^n, \omega^n \rangle$ is a constant function on X, so that $C \in \mathbb{R}$ exists.)

Definition 6.19. Let (X, ω) be Kähler of complex dimension n with induced inner product g. Then $\star : C^\infty(X, \Omega^k_X) \to C^\infty(X < \Omega^{2n-k}_X)$ given by Definition 6.15 on each (T_xX, g_x). Denote the complex linear extension by the same symbol, i.e.

$$\star : C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{2n-k}_{C,\mathbb{C}}.$$

I.e. $\star$ is characterised by the following equation:

$$\alpha \wedge \overline{\star\beta} = \langle \alpha, \beta \rangle \,\mathrm{vol} \quad \text{for} \quad \alpha, \beta \in C^\infty(X, \Omega^k_{X,\mathbb{C}}).$$

Definition 6.20. Let $E \to X$ be a Hermitian vector bundle. We write $C^\infty_c(X, E) := \{s \in C^(X, E) : s \text{ has compact support}\}$. We define the *$L^2$-inner product* as

$$\langle \alpha, \beta \rangle_{L^2} := \int_X \langle \alpha, \beta \rangle \,\mathrm{vol} \quad \text{for} \quad \alpha, \beta in C^\infty_c(X, E).$$

Definition 6.21. Let $E, F \to X$ be Hermitian vector bundles. Let $P : C_c^\infty(X, E) \to C_c^\infty(X, F)$ be $\mathbb{C}$-linear. The *adjoint map* $P^* : C_c^\infty(X, F) \to C_c^\infty(X, E)$ is defined via

$$\langle P\alpha, \beta\rangle_{L^2} = \langle \alpha, P^*\beta\rangle_{L^2} \quad \text{for} \quad \alpha \in C_c^\infty(X, E), \beta \in C_c^\infty(X, F).$$

Lemma 6.22. *Let (X, ω) be Kähler and let $\beta \in C_c^\infty(X, \Omega^{k+1}_{X,\mathbb{C}})$ for some $k \geq 1$. Then*

$$\mathrm{d}^*\beta = - \star \mathrm{d} \star \beta.$$

Proof. Let $\alpha \in C_c^\infty(X, \Omega^k_{X,\mathbb{C}})$. Then

$$\langle \mathrm{d}\alpha, \beta\rangle \operatorname{vol} = \mathrm{d}\alpha \wedge \overline{\star\beta} = \mathrm{d}(\alpha \wedge \overline{\star\beta}) - (-1)^k \alpha \wedge \mathrm{d}\overline{\star\beta}$$

by the Leibniz rule. Hence:

$$\begin{aligned}
\langle \mathrm{d}\alpha, \beta\rangle_{L^2} &= \int_X \langle \mathrm{d}\alpha, \beta\rangle \operatorname{vol} \\
&= (-1)^{k+1} \int_X \alpha \wedge \mathrm{d}\overline{\star\beta} \\
&= \underbrace{(-1)^{k(2n-k)+k+1}}_{=-1} \int_X \alpha \wedge \star \star \mathrm{d}\overline{\star\beta} \\
&= - \int_X \alpha \wedge \overline{\star \star \mathrm{d} \star \beta} \\
&= \langle \alpha, - \star \mathrm{d} \star \beta\rangle_{L^2},
\end{aligned}$$

where in the second step we used Stokes' theorem and in the third step we used equation $(*)$ after Definition 6.15 to work out the correct sign. This proves the claim. $\square$

6.2 Harmonic forms

Definition 6.23. Let (X, ω) be Kähler. The operator

$$\Delta := \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d}$$

is called *Hodge-de Rham operator* or *Laplace operator*. A form $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$ is called *harmonic* if $\Delta\alpha = 0$. We denote

$$\mathcal{H}^k(X) := \{\alpha \in C^\infty(X, \Omega^k_{C,\mathbb{C}}) : \Delta\alpha = 0\}.$$

Lemma 6.24. *Let (X, ω) be Kähler. Let $\alpha \in C_c^\infty(X, \Omega^k_{X,\mathbb{C}})$. Then*

$$\alpha \textit{ is harmonic} \quad \Leftrightarrow \quad \mathrm{d}\alpha = 0 \textit{ and } \mathrm{d}^*\alpha = 0.$$

Proof. "$\Leftarrow$" is clear by definition of Δ.
"$\Rightarrow$": If $\Delta\alpha = 0$, then also $\langle \Delta\alpha, \alpha\rangle_{L^2} = 0$, and therefore

$$\begin{aligned}
0 = \langle \Delta\alpha, \alpha\rangle_{L^2} &= \langle \mathrm{d}^*\,\mathrm{d}\alpha, \alpha\rangle_{L^2} + \langle \mathrm{d}\,\mathrm{d}^*\alpha, \alpha\rangle_{L^2} \\
&= \langle \mathrm{d}\alpha, \mathrm{d}\alpha\rangle_{L^2} + \langle \mathrm{d}^*\alpha, \mathrm{d}^*\alpha\rangle_{L^2}
\end{aligned}$$

and because $\langle \cdot, \cdot\rangle$ is positive definite, we have that $\mathrm{d}\alpha = 0$ and $\mathrm{d}^*\alpha = 0$. $\square$

Exercise 6.25. *Let $X = \mathbb{C}^n$ and*

$$\omega = \frac{i}{2}\sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\overline{z_j}$$

be its standard Kähler form. Writing $z_j = x_j + iy_j$, prove that

$$\Delta f = -\sum_{j=1}^{n}\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right) f \tag{$*$}$$

Remark 6.26. The minus sign in equation $(*)$ makes Δ a non-negative operator. This is the convention typically used in geometry and is called the *geometer's Laplacian.* Some texts define the Laplacian as

$$\Delta f = \sum_{j=1}^{n}\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right) f.$$

This convention is called the *analyst's Laplacian.*

Lemma 6.27.

1. *Δ commutes with $\star$, d, and d^*.*
2. *Δ is formally self-adjoint, i.e.*

$$\langle \Delta\alpha, \beta\rangle_{L^2} = \langle \alpha, \Delta\beta\rangle_{L^2} \quad \textit{for all} \quad \alpha \in C_c^\infty(X, \Omega^k_{X,\mathbb{C}}).$$

3. *Δ is non-negative, i.e. $\langle \Delta\alpha, \alpha\rangle_{L^2} \geq 0$ for all $\alpha \in C_c^\infty(X, \Omega^k_{X,\mathbb{C}})$.*

Proof.

1. We have

$$\begin{aligned}\Delta\star &= (\mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d})\star \\ &= -\underbrace{\mathrm{d}\star\mathrm{d}\star\star}_{=\star\star\mathrm{d}\star\mathrm{d}} - \star\,\mathrm{d}\star\mathrm{d}\star \\ &= \mathrm{d}\Delta,\end{aligned}$$

where we used $\star\star = \pm 1$, where the exact value of the sign does not matter.

Commuting with d and d^* follows directly from the definition of Δ.

2. It is

$$\begin{aligned}\langle \Delta\alpha, \beta\rangle_{L^2} &= \langle \mathrm{d}\,\mathrm{d}^*\alpha, \beta\rangle_{L^2} + \langle \mathrm{d}^*\,\mathrm{d}\alpha, \beta\rangle_{L^2} \\ &= \langle \mathrm{d}^*\alpha, \mathrm{d}^*\beta\rangle_{L^2} + \langle \mathrm{d}\alpha, \mathrm{d}\beta\rangle_{L^2} \\ &= \langle \alpha, \mathrm{d}\,\mathrm{d}^*\beta\rangle_{L^2} = \langle \alpha, \mathrm{d}^*\,\mathrm{d}\beta\rangle_{L^2} \\ &= \langle \alpha, \Delta\beta\rangle_{L^2}.\end{aligned}$$

3. This is the same calculation as in the previous point or as in Lemma 6.24:

$$\langle \Delta\alpha, \alpha\rangle_{L^2} = \langle \mathrm{d}\alpha, \mathrm{d}\alpha\rangle_{L^2} + \langle \mathrm{d}^*\alpha, \mathrm{d}^*\alpha\rangle_{L^2} \geq 0.$$

□

Definition 6.28. For $K \in \{\mathbb{R}, \mathbb{C}\}$ and $U \subset \mathbb{R}^n$, a *local linear differential operator of order* k is a map $P : C^\infty(U, \underline{K^l}) \to C^\infty(U, \underline{K^l})$ of the form

$$P(u) = \sum_{|\alpha| \le k} a_\alpha \cdot \partial^\alpha(u) \tag{$*$}$$

for some $a_\alpha \in C^\infty(U, \underline{K^{l\times l}})$ and where for $\alpha = (\alpha_1, \ldots, \alpha_l)$ we used the notation $\partial^\alpha = \partial^{\alpha_1} \ldots \partial^{\alpha_l}$.
Its *principal symbol at* $x \in U$ $\sigma(P)_x \in \mathbb{R}[\xi_1, \ldots, \xi_n]$ is given by

$$\sigma(P)_x : -\sum_{|\alpha|=k} a_\alpha(x) \cdot \xi^\alpha.$$

I.e. we replaced the symbols ∂^{α_i} by formal variables ξ^{α_i} and obtain a polynomial in these formal variables. Note that for the principal symbol we only consider $|\alpha| = k$, even though the operator P may be defined with lower order parts, i.e. $|\alpha| < k$.
The operator P is called *elliptic*, if $\sigma(P)_x(\xi) \in K^{l\times l}$ is invertible for all $x \in U$ and all $0 \neq \xi \in \mathbb{R}^n$.
Let $E, F \to M$ be vector bundles. A map $P : C^\infty(M, E) \to C^\infty(M, F)$ is called *differential operator of order* k, if in a local trivialisation and local coordinates it is of the form ($*$). (It is easy to check that this is independent of the choice of coordinates, so one may ask that P is of the form ($*$) in *every* local trivialisation and local coordinates.) The operator P is called *elliptic* if its local form is elliptic.

Example 6.29. On $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_{2n})$ we have by Exercise 6.25 that

$$\Delta f = -\sum_{j=1}^{2n} \frac{\partial^2 f}{\partial x_j^2},$$

so its principal symbol is

$$\sigma(\Delta)_x = -\sum_{j=1}^{2n} \xi_j^2$$

for any $x \in \mathbb{R}^{2n}$. Now, for $x \in \mathbb{R}^{2n}$ and $0 \neq \xi \in \mathbb{R}^n$ we have that

$$\sigma(\Delta)_x(\xi) = -\sum_{j=1}^{2n} \xi_j^2 \neq 0 \in \mathbb{R}^{1\times 1},$$

so $\sigma(\Delta)_x(\xi)$ is an invertible (1×1)-matrix, which means that Δ is elliptic.

Proposition 6.30. *Let (X, ω) be Kähler, then $\Delta : C^\infty(X, \Omega^r_{X,\mathbb{C}}) \to C^\infty(X, \Omega^r_{X,\mathbb{C}})$ is elliptic for all $r \ge 0$.*

Note: this proof is different from the one presented in the lecture. In the lecture we used the Weitzenböck formula without proof, below is a proof from first principles. To read this proof, the reminder about the contraction operator $\lrcorner$ from the beginning of Section 6.4 may be helpful.

Proof. **Step 1: Computation in $\mathbb{C}^n$.** Let $\mathbb{C}^n$ be endowed with the standard Kähler form, and write $\mathrm{d}x_1, \ldots, \mathrm{d}x_{2n}$ for its real canonical basis forms. The form an orthonormal basis everywhere, and so we obtain for $f \in C^\infty(\mathbb{C}^n)$:

$$\mathrm{d}\,\mathrm{d}^*(f\,\mathrm{d}x_I) = -\,\mathrm{d}(\operatorname{grad} f \lrcorner\, \mathrm{d}x_I) = -\sum_{k\in I, l\in I^c\cup\{k\}} \frac{\partial^2 f}{\partial x_k \partial x_l}\,\mathrm{d}x_{I\setminus\{k\}\cup\{l\}},$$

$$\mathrm{d}^*\,\mathrm{d}(f\,\mathrm{d}x_I) = \mathrm{d}^*(\mathrm{d}f \wedge \mathrm{d}x_I) = -\sum_{k\in I^c, l\in I\cup\{k\}} \frac{\partial^2 f}{\partial x_k \partial x_l}\,\mathrm{d}x_{I\cup\{k\}\setminus\{l\}},$$

where we used the notation I^c for the complement of I from Definition 6.15. We also used the notations

$$\begin{aligned}
\mathrm{d}x_{J\setminus\{k\}} &= (-1)^{p-1}\,\mathrm{d}x_{j_1}\wedge\cdots\wedge\widehat{\mathrm{d}x_{j_p}}\wedge\cdots\wedge\mathrm{d}x_{j_m} \qquad \text{for} \qquad J=(j_1,\dots,j_m) \text{ and } j_p=k,\\
\mathrm{d}x_{J\cup\{k\}} &= (-1)^{p-1}\,\mathrm{d}x_{j_1}\wedge\cdots\wedge\mathrm{d}x_{j_{p-1}}\wedge\mathrm{d}x_k\wedge\mathrm{d}x_{j_{p+1}}\wedge\cdots\wedge\mathrm{d}x_{j_m}\\
&\qquad \text{for} \qquad J=(j_1,\dots,j_m) \text{ and so that } (j_1,\dots,j_{p-1},k,j_{p+1},\dots,j_m) \text{ are sorted ascendingly.}
\end{aligned}$$

This gives

$$\mathrm{d}x_{I\setminus\{k\}\cup\{l\}} = -\,\mathrm{d}x_{I\cup\{l\}\setminus\{k\}} \qquad \text{for} \qquad k\in I, l\in I^c.$$

Changing the names of the summation indices k and l in the second sum, and writing the summands where $k=l$ extra, we obtain:

$$\begin{aligned}
\Delta(f\,\mathrm{d}x_I) &= \mathrm{d}\,\mathrm{d}^*(f\,\mathrm{d}x_I) + \mathrm{d}^*\,\mathrm{d}(f\,\mathrm{d}x_I)\\
&= -\sum_{k=1}^n \frac{\partial^2 f}{\partial x_k^2}\,\mathrm{d}x_I - \sum_{k\in I, l\in I^c}\frac{\partial^2 f}{\partial x_k\partial x_l}\,\mathrm{d}x_{I\setminus\{k\}\cup\{l\}} + \sum_{l\in I^c, k\in I}\frac{\partial^2 f}{\partial x_l\partial x_k}\,\mathrm{d}x_{I\setminus\{k\}\cup\{l\}}\\
&= \Delta(f)\,\mathrm{d}x_I.
\end{aligned}$$

Step 2: Computation on X. For $p\in X$ let $(U,(x_1,\dots,x_{2n}))$ be a real chart with $x_i(p)=0$ and so that $\frac{\partial}{\partial x_i}(p)$ form an orthonormal basis. We can construct such a frame by starting with an arbitrary chart and then using the Gram-Schmidt procedure on T_pX. That is, $\langle \mathrm{d}x_i, \mathrm{d}x_j\rangle = \delta_{ij} + \mathcal{O}(|x|)$ on all of U. Then

$$\begin{aligned}
\mathrm{d}\,\mathrm{d}^*(f\,\mathrm{d}x_I) &= -\,\mathrm{d}\star\mathrm{d}(f\cdot[\mathrm{d}x_{I^c} + \mathcal{O}(|x|)])\\
&= -\,\mathrm{d}\star(\mathrm{d}f\wedge[\mathrm{d}x_{I^c}+\mathcal{O}(|x|)]) + \text{l.o.t.}\\
&= -\sum_{k\in I, l\in I^c\cup\{k\}}\frac{\partial^2 f}{\partial x_k\partial x_l}\left[\mathrm{d}x_{I\setminus\{k\}\cup\{l\}} + \mathcal{O}(|x|)\right] + \text{l.o.t.} \qquad \text{and similarly}\\
\mathrm{d}^*\,\mathrm{d}(f\,\mathrm{d}x_I) &= -\sum_{k\in I^c, l\in I\cup\{k\}}\frac{\partial^2 f}{\partial x_k\partial x_l}\left[\mathrm{d}x_{I\cup\{k\}\setminus\{l\}} + \mathcal{O}(|x|)\right] + \text{l.o.t.},
\end{aligned}$$

where *l.o.t.* stands for *lower order terms* and denotes terms involving at most first derivatives of f. These terms do not matter when computing the principal symbol of an operator. Thus, in the point p we have

$$\Delta(f\,\mathrm{d}x_I)(p) = (\Delta f)(p)\cdot\mathrm{d}x_I(p) + \text{l.o.t.}$$

Now let $\mathrm{d}x_{I_1},\dots,\mathrm{d}x_{I_d}$ be a local trivialisation of Ω^r. I.e. $d=\operatorname{rank}(\Omega^r) = \binom{2n}{r}$. In this trivialisation we therefore have for the Laplacian on r-forms, temporarily denoted as Δ^r, and $\xi\in\mathbb{R}^{2n}$:

$$(\sigma\Delta^r)_x(\xi) = \begin{pmatrix} (\sigma\Delta)_x(\xi) & 0 & \dots & 0\\ 0 & (\sigma\Delta)_x(\xi) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & (\sigma\Delta)_x(\xi)\end{pmatrix} \in \mathbb{R}^{d\times d},$$

which is invertible by Example 6.29. □

Theorem 6.31 (Fredholm Alternative). *Let $E, F\to X$ be vector bundles over a complex manifold. Let $P: C^\infty(X,E)\to C^\infty(X,F)$ be a linear differential operator of order k. Then*

1. $\operatorname{Ker} P^*$ *is finite dimensional.*

2. *We have the decompositions*

$$\begin{aligned} C^\infty(X,F) &= \operatorname{Im} P \oplus \operatorname{Ker} P^* = P(C^\infty(X,E)) \oplus \operatorname{Ker} P^* \\ L^2(X,F) &= P(L^2(X,E)) \oplus \operatorname{Ker} P^*. \end{aligned}$$

We omit its proof here, see e.g. [2] for the statement and references that contain a proof.

Theorem 6.32 (Hodge Theorem). *Let (X,ω) be a compact Kähler manifold. Then, for $k \geq 0$ we have*

$$\begin{aligned} C^\infty(X,\Omega^k_{X,\mathbb{C}}) &= \mathcal{H}^k(X) \oplus \Delta(C^\infty(X,\Omega^k_{X,\mathbb{C}}) && (*) \\ &= \mathcal{H}^k(X) \oplus \mathrm{d}(C^\infty(X,\Omega^{k-1}_{X,\mathbb{C}})) \oplus \mathrm{d}^*(C^\infty(X,\Omega^{k+1}_{X,\mathbb{C}})0 && (**) \end{aligned}$$

where all decompositions are orthogonal.

Proof. **Step 1: Orthogonalities**. We first check $\operatorname{Im}\mathrm{d} \perp \operatorname{Im}\mathrm{d}^*$: for $\sigma \in C^\infty(X,\Omega^{k-1}_{X,\mathbb{C}})$ and $\tau \in C^\infty(X,\Omega^{k+1}_{X,\mathbb{C}})$ we have

$$\langle \mathrm{d}\sigma, \mathrm{d}^*\tau\rangle_{L^2} = \langle \mathrm{d}\,\mathrm{d}\sigma, \tau\rangle_{L^2} = 0$$

because $\mathrm{d}\,\mathrm{d} = 0$. The other orthogonalities follow analogously because harmonic implies closed and co-closed by Lemma 6.24.
Step 2: Direct sum decompositions. Equation $(*)$ is precisely Theorem 6.31 together with $\Delta = \Delta^*$ (see Lemma 6.27).
To see $(**)$: $\operatorname{Im}\Delta \subset \operatorname{Im}\mathrm{d} \oplus \operatorname{Im}\mathrm{d}^*$ follows from the definition of Δ. To see the converse inclusion, let $\sigma \in \operatorname{Im} d \oplus \operatorname{Im}\mathrm{d}^*$. By part 1, we have that $\sigma \perp \operatorname{Ker}\Delta$, so by $(*)$ we have $\sigma \in \operatorname{Im}\Delta$, which proves the claim. □

Definition 6.33. For $\alpha \in C^\infty(X,\Omega^k_{X,\mathbb{C}})$ we have the decomposition $\alpha = \Delta\beta + \mathrm{proj}_{\mathcal{H}}(\alpha)$ for some $\beta \perp \operatorname{Ker}\Delta$ by Theorem 6.32. Here, $\mathrm{proj}_{\mathcal{H}}$ denotes the L^2-orthogonal projection

$$\mathrm{proj}_{\mathcal{H}} : C^\infty(X,\Omega^k_{X,\mathbb{C}}) \to \mathcal{H}^k(X).$$

The map

$$\begin{aligned} G : C^\infty(X,\Omega^k_{X,\mathbb{C}}) &\to (\operatorname{Ker}\Delta)^\perp \subset C^\infty(X,\Omega^k_{X,\mathbb{C}}) \\ \alpha &\mapsto \beta \end{aligned}$$

is called *Green's operator.*

Lemma 6.34. *The map G commutes with* d, d^* *and* $\star$.

Proof. We give the proof for d, the other statements follow analogously. (If one were to prove the statement only for d the proof could be shortened, but we write it in such a way that it can easily be adapted to the other operators.)
We have $\mathrm{d}(\mathcal{H}^k(X)^\perp) \subset \mathcal{H}^{k+1}(X)^\perp$ because for $\sigma \in \mathcal{H}^k(X)^\perp$ and $\tau \in \mathcal{H}^{k+1}(X)$:

$$\langle \mathrm{d}\sigma, \tau\rangle = \langle \sigma, \mathrm{d}^*\tau\rangle = 0$$

where we used Lemma 6.24 in the second step. Also, $\mathrm{d}(\mathcal{H}^k(X)) \subset \mathcal{H}^{k+1}(X)$ by Lemma 6.27.
Thus, for $\alpha = \mu + \nu$, where $\mu \in \mathcal{H}^k(X)$ and $\nu \in (\mathcal{H}^k(X))^\perp$ we have $\mathrm{d}\alpha = \mathrm{d}\mu + \mathrm{d}\nu$ with $\mathrm{d}\mu \in \mathcal{H}^{k+1}(X)$ and $\mathrm{d}\nu \in (\mathcal{H}^{k+1}(X))^\perp$, so $\mathrm{d}(\mathrm{proj}_{\mathcal{H}}(\alpha)) = \mathrm{proj}_{\mathcal{H}}(\mathrm{d}\alpha)$.
Hence, $\mathrm{d}\alpha = \mathrm{d}(\Delta\beta + \mathrm{proj}_{\mathcal{H}}(\alpha)) = \Delta(\mathrm{d}\beta) + \mathrm{proj}_{\mathcal{H}}(\mathrm{d}\alpha)$, and therefore $G(\mathrm{d}\alpha) = \mathrm{d}\beta = \mathrm{d}(G(\alpha))$. □

Corollary 6.35. *Let (X, ω) be a compact Kähler manifold. Then for $k \geq 0$, the map*

$$\begin{aligned} F : \mathcal{H}^k(X) &\to H^k(X, \mathbb{C}) \\ \alpha &\mapsto [\alpha] \end{aligned}$$

is an isomorphism.

Proof. **Surjectivity**: Let $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$, then

$$\begin{aligned} \alpha &= \Delta G\alpha + \operatorname{proj}_{\mathcal{H}} \alpha \\ &= \mathrm{d}\,\mathrm{d}^*(G\alpha) + \mathrm{d}^*\,\mathrm{d}(G\alpha) + \operatorname{proj}_{\mathcal{H}} \alpha \\ &= \mathrm{d}\,\mathrm{d}^*(G\alpha) + \mathrm{d}^*(G(\mathrm{d}\alpha)) + \operatorname{proj}_{\mathcal{H}} \alpha. \end{aligned}$$

If α represents a de Rham cohomology class, then it is closed, i.e. $\mathrm{d}^*(G(\mathrm{d}\alpha)) = 0$, and hence

$$\alpha = \mathrm{d}\,\mathrm{d}^*(G\alpha) + \operatorname{proj}_{\mathcal{H}} \alpha,$$

so $[\alpha] = [\operatorname{proj}_{\mathcal{H}} \alpha] = F(\operatorname{proj}_{\mathcal{H}} \alpha)$.
Injectivity: Let $[\alpha] = 0$, i.e. $\alpha \in \operatorname{Im} \mathrm{d}$, for some $\alpha \in \mathcal{H}^k(X)$. By $(**)$ of Theorem 6.32 we have that $\alpha = 0$. $\square$

Theorem 6.36 (Poincaré duality). *Let (X, ω) be a compact Kähler manifold of complex dimension n. Then*

$$\begin{aligned} \star : H^k(X, \mathbb{C}) &\to H^{2n-k}(X, \mathbb{C}) \\ [\omega] \text{ with } \omega \text{ being harmonic} &\mapsto [\star\omega] \end{aligned}$$

is an isomorphism.

Proof. If ω is harmonic, then by Lemma 6.24 we have $\mathrm{d}(\star\omega) = 0$, so the map is well-defined.
By Lemma 6.27, we have that $\star$ commutes with Δ, so $\star : \mathcal{H}^k(X) \to \mathcal{H}^{2n-k}(X)$ is an isomorphism. By Corollary 6.35, we have that $H^k(X, \mathbb{C}) \cong \mathcal{H}^k(X)$, hence the map from the theorem statement is an isomorphism. $\square$

6.3 Harmonic (p, q)-forms

Let (X, ω) be a Kähler manifold of complex dimension n and $x_0 \in X$. We can choose coordinates $z_1, \ldots, z_n$ around x_0 such that $\mathrm{d}z_1, \ldots, \mathrm{d}z_n$ is a local frame of $\Omega^{1,0}_{X,\mathbb{C}}$ satisfying

$$\langle \mathrm{d}z_i(x_0), \mathrm{d}z_j(x_0) \rangle = \delta_{ij}.$$

(We can only arrange for this to hold in a single point, not necessarily in a neighbourhood of that point.)
Thus, for two (p, q)-forms η^1 and η^2 given as

$$\eta^j = \sum_{|I|=p, |J|=q} \eta^j_{I,J}\, \mathrm{d}z_I \wedge \mathrm{d}\overline{z_J} \quad \text{for} \quad j \in \{1, 2\}$$

we have

$$\langle \eta^1(x_0), \eta^2(x_0) \rangle = \sum_{|I|=p, |J|=q} \eta^1_{I,J}(x_0) \cdot \overline{\eta^2_{I,J}(x_0)}. \tag{$*$}$$

Lemma 6.37. *The map*

$$\star : C^\infty(X, \Omega^{p,q}_X) \to C^\infty(X, \Omega^{n-q,n-p}_X)$$

is a $\mathbb{C}$-linear isometry.

Proof. Let $\eta \in C^\infty(X, \Omega_X^{p,q})$, $\eta \neq 0$, then

$$\eta \wedge \overline{\star\eta} = \langle \eta, \eta \rangle \operatorname{vol} \neq 0 \in C^\infty(X, \Omega_X^{n,n}).$$

The only way the wedge product with a (p,q)-form on the left can be non-zero, is if $\overline{\star\eta} \in C^\infty(X, \Omega^{n-p,n-q})$, and hence $\star\eta \in C^\infty(U, \Omega_X^{n-q,n-p})$. Thus, $\star$ maps indeed between the two spaces of sections claimed in the statement of the lemma.
It remains to show that it defines an isometry:

$$\langle \star\eta, \star\eta \rangle \operatorname{vol} = \star\eta \wedge \overline{\star \star \eta} = \pm \star \eta \wedge \overline{\eta} = \overline{\eta} \wedge \star\eta = \langle \overline{\eta}, \overline{\eta} \rangle \operatorname{vol} = \langle \eta, \eta \rangle \operatorname{vol},$$

where in the second and third step we used that the signs introduced by $\star\star$ and swapping the order of the wedge product cancel out, and in the last step we used $\langle \eta, \eta \rangle \in \mathbb{R}$, so this number is unchanged by conjugation. □

We have the adjoint operators

$$\partial^* : C_c^\infty(X, \Omega_X^{p+1,q}) \to C_c^\infty(X, \Omega_X^{p,q}),$$
$$\overline{\partial}^* : C_c^\infty(X, \Omega_X^{p,q}) \to C_c^\infty(X, \Omega_X^{p,q-1})$$

and the following is proved analogously to Lemma 6.22:

Lemma 6.38. *We have $\partial^* = - \star \overline{\partial} \star$ and $\overline{\partial}^* = - \star \partial \star$.*

Definition 6.39. The corresponding Laplace operators are defined as $\Delta_\partial := \partial\partial^* + \partial^*\partial$ and $\Delta_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. A form α is called *∂-harmonic* if $\Delta_\partial \alpha = 0$ and is called *$\overline{\partial}$-harmonic* if $\Delta_{\overline{\partial}}\alpha = 0$.

The following is then analogue to Lemma 6.24:

Lemma 6.40. *Let (X, ω) be Kähler, then for $\alpha \in C_c^\infty(X, \Omega_X^{p,q})$:*

1. *α is Δ_∂-harmonic if and only if $\partial\alpha = \partial^*\alpha = 0$,*
2. *α is $\Delta_{\overline{\partial}}$-harmonic if and only if $\overline{\partial}\alpha = \overline{\partial}^*\alpha = 0$.*

Definition 6.41. We write

$$\mathcal{H}^{p,q}(X) := \{\alpha \in C^\infty(X, \Omega_X^{p,q}) : \Delta_{\overline{\partial}}\alpha = 0\}$$

for the set of $\overline{\partial}$-harmonic (p,q)-forms.

6.4 Lefschetz operator and Kähler identities

Definition 6.42. Let (X, ω) be Kähler. The map

$$L : C_c^\infty(X, \Omega_{X,\mathbb{C}}^k) \to C_c^\infty(X, \Omega_{X,\mathbb{C}}^{k+2})$$

is called *Lefschetz operator*, and its operator is denoted as

$$\Lambda = L^* : C_c^\infty(X, \Omega_{X,\mathbb{C}}^{k+2}) \to C_c^\infty(X, \Omega_{X,\mathbb{C}}^k).$$

One easily checks that Λ has the formula $\Lambda = (-1)^k \star L \star$.
We quickly recall the contraction operator between vector fields and differential forms. For $X \in C^\infty(X, T_{X,\mathbb{C}})$ and $\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}^k)$ we define the *contraction* $X \lrcorner \alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}^{k-1})$ as:

$$X \lrcorner \alpha(v_1, \ldots, v_{k-1}) = \alpha(X, v_1, \ldots, v_{k-1}) \quad \text{for} \quad v_1, \ldots, v_{k-1} \in C^\infty(X, T_{X,\mathbb{C}}).$$

For example, we have

$$\frac{\partial}{\partial z_m} \lrcorner \, \mathrm{d}z_I = \begin{cases} 0 & \text{if } m \notin I \\ (-1)^{l-1} \, \mathrm{d}z_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}z_{i_l}} \wedge \cdots \wedge \mathrm{d}z_{i_k} & \text{if } m = i_l \end{cases}$$

and

$$v \lrcorner (\alpha \wedge \beta) = (v \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (v \lrcorner \beta). \tag{$*$}$$

Example 6.43. Let $U \subset \mathbb{C}^n$ be open and let

$$\omega = \frac{i}{2} \sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\overline{z_j}$$

be the standard Kähler form. Then $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ is a holomorphic orthonormal basis of $T^{1,0}U$. For $\alpha \in C_c^\infty(U, \Omega_U^{p,q})$ write

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{I,J} \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z_J}.$$

Then:

$$\begin{aligned}
\overline{\partial}^* \alpha &= - \star \partial \star \left(\sum_{|I|=p,|J|=q} \alpha_{I,J} \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z_J} \right) \\
&= - \star \partial \left(\sum_{I,J} \mathrm{d}\overline{z}_{I^c} \wedge \mathrm{d}z_{J^c} \right) \\
&= - \star \left(\sum_{k=1}^{n} \sum_{I,J} \frac{\partial \alpha_{I,J}}{\partial z_k} \, \mathrm{d}z_k \wedge \mathrm{d}\overline{z_{I^c}} \wedge \mathrm{d}z_{J^c} \right) \\
&= - \sum_{k=1}^{n} \sum_{I,J} \frac{\partial \alpha_{I,J}}{\partial z_k} \cdot \frac{\partial}{\partial \overline{z_k}} \lrcorner \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z_J} \\
&=: \sum_{k=1}^{n} \frac{\partial}{\partial \overline{z_k}} \lrcorner \left(\frac{\partial}{\partial z_k} \alpha \right)
\end{aligned}$$

Lemma 6.44. *Let $U \subset \mathbb{C}^n$ be open and*

$$\omega = i \sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\overline{z_j},$$

then $[\overline{\partial}^*, L] = i\partial$.

Proof. We want to check the equality of operators acting on some element $\alpha \in C^\infty(U, \Omega^k_{U,\mathbb{C}})$. By linearity it suffices to check $\alpha = \alpha_{I,J} \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z_J}$. Now

$$\begin{aligned}
[\overline{\partial}^*, L]\alpha &= - \sum_{k=1}^{n} \frac{\partial}{\partial \overline{z_k}} \lrcorner \underbrace{\frac{\partial}{\partial z_k} (\omega \wedge \alpha)}_{= \omega \wedge \frac{\partial}{\partial z_k} \alpha} - \omega \wedge \left(- \sum_{k=1}^{n} \frac{\partial}{\partial \overline{z_k}} \lrcorner \frac{\partial}{\partial z_k} \alpha \right) \\
&= - \sum_{k=1}^{n} \left(\frac{\partial}{\partial \overline{z_k}} \lrcorner \omega \right) \wedge \frac{\partial}{\partial z_k} \alpha - \omega \wedge \left(\frac{\partial}{\partial \overline{z_k}} \lrcorner \frac{\partial}{\partial z_k} \alpha \right) + \omega \wedge \left(\sum_{k=1}^{n} \frac{\partial}{\partial \overline{z_k}} \lrcorner \frac{\partial}{\partial z_k} \alpha \right) \\
&= i \sum_{k=1}^{n} \mathrm{d}z_k \wedge \frac{\partial}{\partial z_k} \alpha \\
&= i \partial \alpha
\end{aligned}$$

where in the first step we used Example 6.43; we used equation $(*)$ from before Example 6.43 in the second step; and we used $\frac{\partial}{\partial \overline{z_k}} \lrcorner \omega = -i \, \mathrm{d}z_k$ in the third step. $\square$

This is a result on $\mathbb{C}^n$ and we now need to transport it to an arbitrary Kähler manifold.

Theorem 6.45 (Existence of Normal Coordinates). *Let (X, ω) be Kähler and $x \in X$. Then there exist complex coordinates $z_1, \ldots, z_n$ around x such that if we write the Kähler form in these coordinates as*

$$\omega = i \sum_{j,k=1}^{n} h_{j,k} \, \mathrm{d}z_j \wedge \mathrm{d}\overline{z_k},$$

then $h_{j,k} = \delta_{jk} + O(|z|^2)$.

Proof. As before, we can find holomorphic local coordinates $z_1, \ldots, z_n$ around x such that $dz_1, \ldots, dz_n$ is a frame of $(T^{1,0}X)^*$ which is orthonormal at x and such that $z_i(x) = 0$ for $i \in \{1, \ldots, n\}$. Thus, we may write

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\overline{z_k}$$

where $h_{j,k} = \delta_{j,k} + O(|z|)$. Thus, there exist $a_{jkl}, a'_{jkl} \in \mathbb{C}$ such that

$$h_{j,k} = \delta_{j,k} + \sum_{l=1}^{n} \left(a_{jkl} z_l + a'_{jkl} \overline{z_l} \right) + O\left(|z|^2 \right). \tag{$*$}$$

Since $(h_{j,k})$ is Hermitian, we have

$$\overline{a_{kjl}} = a'_{jkl} \qquad \text{for any } j, k, l. \tag{$**$}$$

Since ω is closed, we have

$$a_{jkl} = a_{lkj} \qquad \text{for any } j, k, l. \tag{$***$}$$

Define

$$\xi_k = z_k + \frac{1}{2} \sum_{j,l=1}^{n} a_{jkl} z_j z_l \qquad k = 1, \ldots, n.$$

Then, by the Inverse Function Theorem (cf. Corollary 2.19), $\xi_1, \ldots, \xi_n$ are local holomorphic coordinates and, by $(***)$, we have

$$\begin{aligned} d\xi_k &= dz_k + \frac{1}{2} \sum_{j,l=1}^{n} a_{jkl} \left(z_j dz_l + z_l dz_j \right) \\ &= dz_k + \frac{1}{2} \sum_{j,l=1}^{n} \left(a_{jkl} + a_{lkj} \right) z_l dz_j \\ &= dz_k + \sum_{j,l=1}^{n} a_{jkl} z_l dz_j. \end{aligned}$$

Thus,

$$i \sum_{k=1}^{n} d\xi_k \wedge d\overline{\xi_k} = i \sum_{k=1}^{n} dz_k \wedge d\overline{z_k} + i \sum_{j,k,l=1}^{n} \left(\overline{a_{jkl} z_l} dz_k \wedge d\overline{z_j} + a_{jkl} z_l dz_j \wedge d\overline{z_k} \right) + O\left(|z|^2 \right)$$

By $(**)$, we have

$$\sum_{j,k,l=1}^{n} \overline{a_{jkl} z_l} dz_k \wedge d\overline{z_j} = \sum_{j,k,l=1}^{n} \overline{a_{kjl} z_l} dz_j \wedge d\overline{z_k} = \sum_{j,k,l=1}^{n} a'_{jkl} \overline{z_l} dz_j \wedge d\overline{z_k}.$$

Thus,

$$i\sum_{k=1}^{n} d\xi_k \wedge \overline{d\xi_k} = i\sum_{j,k=1}^{n}\left(\delta_{j,k} + \sum_{l=1}^{n} a_{jkl}z_l + a'_{jkl}\overline{z_l}\right) dz_j \wedge d\overline{z_k} + O\left(|z|^2\right) = \omega + O\left(|z|^2\right).$$

For small $|z|$ we have that $|z|^2 \leq \frac{1}{2}|z|$, so $O(|z|^2) = O(|\xi|^2)$, and therefore the theorem holds in the coordinates ξ_i. □

Theorem 6.46 (Kähler identities). *Let (X,ω) be a Kähler manifold.*
Then

1. $[\overline{\partial}^*, L] = i\partial$
2. $[\partial^*, L] = -i\overline{\partial}$
3. $[\Lambda, \overline{\partial}] = -i\partial^*$
4. $[\Lambda, \partial] = i\overline{\partial}^*$.

Proof of Theorem 6.46. We first prove (1). We have

$$\bar{\partial}^* = -\star\,\partial\,\star\,.$$

Therefore, in the normal coordinates from Theorem 6.45 around $x \in X$ we have that $(\overline{\partial}^*\alpha)(x)$ has the same formula as on $\mathbb{C}^n$, and the claim follows from Lemma 6.44.
We now prove (3). Let α and β be (p,q)-forms, then by (1) we have

$$([\Lambda, \overline{\partial}]\alpha, \beta) = (\alpha, [\overline{\partial}^*, L]\beta) = (\alpha, i\partial\beta) = (-i\alpha, \partial\beta) = (-i\partial^*\alpha, \beta).$$

Thus, (3) follows.

Since $\overline{L} = L$, we have that (1) implies (2) and, since $\overline{\Lambda} = \Lambda$, we have that (3) implies (4). □

Theorem 6.47. *Let (X,ω) be a Kähler manifold.*
Then

$$\Delta = 2\Delta_\partial = 2\Delta_{\overline{\partial}}.$$

Proof. Since $d = \partial + \overline{\partial}$, we have

$$\Delta = (\partial + \bar{\partial})\left(\partial^* + \bar{\partial}^*\right) + \left(\partial^* + \bar{\partial}^*\right)(\partial + \bar{\partial}).$$

By (4) of Theorem 6.46, we have

$$\overline{\partial}^* = -i[\Lambda, \partial] = -i\Lambda\partial + i\partial\Lambda.$$

Thus, since $\partial^2 = 0$, we have

$$(\partial + \bar{\partial})\left(\partial^* + \bar{\partial}^*\right) = \partial\partial^* + \bar{\partial}\partial^* - i\partial\Lambda\partial - i\bar{\partial}\Lambda\partial + i\bar{\partial}\partial\Lambda.$$

Similarly, we have

$$\left(\partial^* + \bar{\partial}^*\right)(\partial + \bar{\partial}) = \partial^*\partial + \partial^*\bar{\partial} + i\partial\Lambda\partial + i\partial\Lambda\bar{\partial} - i\Lambda\partial\bar{\partial}.$$

By (3) of Theorem 6.46, we have

$$\partial^*\overline{\partial} = i[\Lambda, \overline{\partial}]\,\overline{\partial} = -i\overline{\partial}\Lambda\overline{\partial} = -i\overline{\partial}[\Lambda, \overline{\partial}] = -\overline{\partial}\partial^*.$$

Thus, using (3) of Theorem 6.46 again, we obtain

$$\begin{aligned}\Delta &= \partial\partial^* - i\bar\partial\Lambda\partial + i\bar\partial\partial\Lambda + \partial^*\partial + i\partial\Lambda\bar\partial - i\Lambda\partial\bar\partial \\ &= \Delta_\partial - i\Lambda\partial\bar\partial - i\bar\partial\Lambda\partial + i\partial\Lambda\bar\partial + i\bar\partial\partial\Lambda \\ &= \Delta_\partial + i\Lambda\partial\bar\partial - i\bar\partial\Lambda\partial + i\partial\Lambda\bar\partial - i\partial\bar\partial\Lambda \\ &= \Delta_\partial + i([\Lambda,\bar\partial]\partial + \partial[\Lambda,\bar\partial]) \\ &= \Delta_\partial + i(-i\partial^*\partial - i\partial\partial^*) = 2\Delta_\partial.\end{aligned}$$

Therefore, the first equality follows.
The second equality follows by similar calculations. ☐

6.5 Hodge decomposition

Theorem 6.48. *Let (X,ω) be a compact Kähler manifold. For $p,q \geq 0$ we have that*

1. *$\mathcal{H}^{p,q}(X)$ is finite dimensional.*
2. *We have the following orthogonal decompositions:*

$$\begin{aligned}C^\infty(X,\Omega_X^{p,q} &= \mathcal{H}^{p,1}(X) \oplus \Delta_{\overline\partial}(C^\infty(X,\Omega_X^{p,q})) && (*)\\ &= \mathcal{H}^{p,q}(X) \oplus \overline\partial C^\infty(X,\Omega_X^{p,q-1}) \oplus \overline\partial^* C^\infty(X,\Omega_X^{p,q+1}), && (**)\\ \operatorname{Ker}\overline\partial &= \mathcal{H}^{p,q}(X) \oplus \overline\partial(C^\infty(X,\Omega_X^{p,q-1})), && (***)\\ \operatorname{Ker}\overline\partial^* &= \mathcal{H}^{p,q}(X) \oplus \overline\partial^* C^\infty(X,\Omega_X^{p,q+1}).\end{aligned}$$

Proof. We have $\Delta = 2\Delta_{\overline\partial}$ by Theorem 6.47, so $\Delta_{\overline\partial}$ is elliptic and $(*)$ is the Fredholm alternative for $\Delta_{\overline\partial}$, Theorem 6.31. $(**)$ is proved exactly as Theorem 6.32. It remains to check $(***)$:
"$\supset$" follows from $\overline{\partial\partial} = 0$ and Lemma 6.40.
"$\subset$": let $\alpha \in C^\infty(X,\Omega_X^{p,q})$ with $\overline\partial\alpha = 0$. Write $\alpha = \alpha_{\mathcal{H}} + \overline\partial(\sigma) + \overline\partial^*(\tau)$. Then $\overline\partial\alpha = 0$ gives $\overline{\partial\partial}^*\tau = 0$, and therefore

$$0 = \langle\overline{\partial\partial}^*\tau,\tau\rangle_{L^2} = \langle\overline\partial^*\tau,\overline\partial^*\tau\rangle_{L^2}\rangle,$$

thus $\overline\partial^*\tau = 0$, so $\alpha \in \mathcal{H}^{p,q}(X) \oplus \overline\partial(C^\infty(X,\Omega_X^{p,q-1}))$.
The last statement is proved analogously, and one checks that the decompositions are orthogonal as in Theorem 6.32. ☐

Corollary 6.49. *Let (X,ω) be Kähler. Then, for $p,q \geq 0$, we have*

$$\mathcal{H}^{p,q}(X) \cong H^{p,q}(X).$$

Proof. Analogous to Corollary 6.35. ☐

Lemma 6.50. *Let (X,ω) be Kähler. Then Δ maps (p,q)-forms to (p,q)-forms.*

Proof. The operator Δ_∂ maps (p,q)-forms to (p,q)-forms by definition. By Theorem 6.47 we have $\Delta = 2\Delta_\partial$, which proves the claim. ☐

Theorem 6.51. *Let (X,ω) be Kähler. For $k \geq 0$ we have*

$$\mathcal{H}^k(X) = \oplus_{p+q=k}\mathcal{H}^{p,q}(X) \qquad (*)$$

and for $p,q \geq 0$ we have

$$\mathcal{H}^{p,q}(X) = \overline{\mathcal{H}^{q,p}(X)}. \qquad (**)$$

Proof. Any element $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$ can be decomposed as $\alpha = \sum \alpha_{p,q}$ with $\alpha_{p,q} \in C^\infty(X, \Omega^{p,q}_{X,\mathbb{C}})$. Then by Lemma 6.50 we have that α is harmonic if and only if $\alpha_{p,q}$ are harmonic for all p, q. This proves $(*)$.
For $\beta \in C^\infty(X, \Omega^{p,q}_X)$ we have

$$\Delta\beta = 0 \Leftrightarrow \Delta_\partial\beta = 0 \Leftrightarrow \Delta_{\overline{\partial}}\overline{\beta} = 0 \Leftrightarrow \Delta\overline{\beta} = 0,$$

where we used Theorem 6.47 for the first and last equivalence and we applied complex conjugation to both sides for the second equivalence. This proves $(**)$. □

Theorem 6.52 (Hodge decomposition theorem). *Let (X, ω) be a compact Kähler manifold. Then*

$$H^k(X, \mathbb{C}) = \bigoplus_{k=p+q} H^{p,q}(X) \quad \text{and} \quad H^{p,q}(X) = \overline{H^{p,q}(X)}.$$

Proof. This is a combination of the following statements:

$$\begin{aligned} &\text{Theorem 6.51}: && \mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X), \\ &\text{Corollary 6.49}: && \mathcal{H}^{p,q}(X) = H^{p,q}(X), \\ &\text{Corollary 6.35}: && \mathcal{H}^k(X) = H^k(X, \mathbb{C}). \end{aligned}$$

□

We can now give a proof of *Serre duality* (see Theorem 4.41) *in the Kähler case.* The statement is true on any compact complex manifold, even if no Kähler structure exists. It is proved by choosing an arbitrary Riemannian metric (i.e. positive definite inner product on the tangent space), and then using its Hodge star $\star$. However, our proof really uses the Kähler property and the generalisation is not easy.

Proof of Theorem 4.41 in the Kähler case. The isomorphism is given by

$$\begin{aligned} H^{p,q}(X) &\to H^{n-p,n-q}(X)^* \\ \alpha &\mapsto \langle \bullet, \overline{\star\alpha} \rangle_{L^2}. \end{aligned}$$

It is an isomorphism by Theorem 6.52, the first point of Lemma 6.27, and the fact that for any Hilbert space $(H, \langle \cdot, \cdot \rangle)$ the map

$$\begin{aligned} H &\to H^* \\ v &\mapsto \langle \bullet, v \rangle \end{aligned}$$

is a complex anti-linear isomorphism. This last fact is the *Riesz representation theorem.* We only need the finite-dimensional version for vector space $H^{p,q}(X)$ though, which is very easy to prove. (The proof of the Riesz representation theorem is not easy for infinite dimensional spaces.) □

The Hodge numbers of a compact Kähler manifold are usually displayed as a *Hodge diamond*, see Fig. 4.
The symmetries within the Hodge diamond are:

1. *Conjugation*, i.e. $h^{p,q} = h^{q,p}$ (Theorem 6.52);
2. *Hodge star* $\star : H^{p,q}(X) \to H^{n-q,n-p}(X)$ (Point 1 of Lemma 6.27);
3. *Serre duality*, which is not an additional symmetry, but just the composition of $\star$ and conjugation (Theorem 4.41).

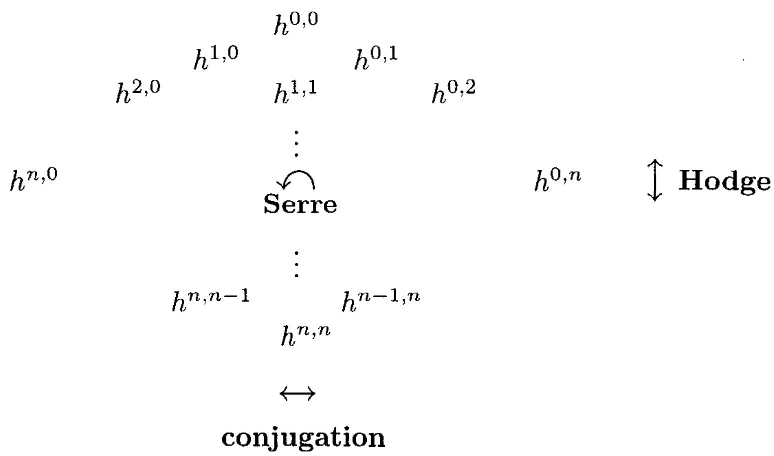

Figure 4: Hodge diamond and its symmetries.

7 Epilogue

Here are some areas research areas in complex geometry:

1. *Gauge theory*: this was mentioned at the end of Section 5.5. In pure maths the goal is to study Hermite-Einstein connections or stable bundles (which in a precise sense are equivalent) to define numerical invariants of Kähler manifolds (or complex manifolds). Many of these equations are motivated from physics, where solutions are needed to model physical phenomena. A currently popular gauge theoretic equation is the *Hull-Strominger system*, for which not many solutions are known. See e.g. for [4] for an introduction for mathematicians, which contains the references to the original physics literature.

2. *Calabi-Yau manifolds*: A compact Kähler manifold (X, ω) is called *Calabi-Yau* if $c_1(K_X) = 0$, where $K_X = \bigwedge^{\dim M}(T^{1,0}X)^*$ is the canonical bundle. This condition is quite easy to check, but it has a surprising implication: by Yau's proof of the Calabi conjecture (a modern presentation of the proof can be found in [8]), X admits a Riemannian metric that is Ricci-flat. There are not many ways known to construct Ricci-flat manifolds, and the largest supply comes from Calabi-Yau manifolds. There are many exciting research directions in this area:

   - Generalisation to non-compact manifolds: this was pioneered by Tian and Yau in [13], but since then the results have been refined and generalised in many ways.
   - Generalisation to singular spaces: by now it is reasonably well understood which singular spaces admit singular Calabi-Yau metrics. Roughly speaking, a singular Kähler manifold with trivial canonical bundle admits a (singular) Calabi-Yau metric if and only if its singularities are *(Kawamata) log-terminal.* [6] is a reference that is well readable for differential geometers, though this is not as general as possible.
   - What does the metric look like? Yau's theorem is only an abstract existence results and does not give a description of the metric. It is of massive interest in Physics to approximate Calabi-Yau

metrics with computer aids and then calculate things with respect to this approximate Calabi-Yau metric, see e.g. [5]. Without computer assistance, one can often describe Calabi-Yau metrics in special situations using *gluing constructions*. An example is [12] and the recent proof of the weak SYZ conjecture surveyed in [9] which is a milestone in this field. [6] also falls into this category to some extent.

- Kähler-Einstein metrics: Ricci-flat manifolds are great, but there are analogues of the Calabi conjecture for Kähler manifolds and metrics with positive or negative Einstein constant. The detailed statements are much more complicated and involve an interesting stability condition, much like in our definition of stable bundle. See [3].

3. *Enumerative geometry*: on a compact Kähler manifold (X, ω) one can define the Gromov-Witten invariant, which is roughly speaking a count of the number of one-dimensional complex submanifolds of X in a fixed cohomology class.

   Extending this to non-compact manifolds is an active research area.

   Also, instead of counting complex curves, there are proposals to count other examples of calibrated submanifolds in manifolds that needn't be complex, but achieving this is still a long way off.

4. *Geometric flows:* the two most studied geometric flows are:

   - the Ricci flow on a Riemannian manifold: $\frac{\partial}{\partial t} g_t = -2 \operatorname{Ric}_{g_t}$,
   - the mean curvature flow of a submanifold $i : M_t \to N$: $\frac{\partial}{\partial t} M_t(x) = \vec{H}(M_t(x))$.

   Studying these flows requires hard analysis, but on Kähler manifolds two miracles happen which make studying these flows much easier:

   - Ricci flow preservers the Kähler property,
   - Mean curvature flow preserves the *Lagrangian* property, i.e. $\omega|_{M_t} = 0$.

   Thanks to this, the formation of singularities under these flows on Kähler manifolds is much better understood than the general case, at least conjecturally.

5. *Moduli spaces*: let e.g. $\mathscr{M} = \{X : \text{compact Calabi-Yau of cx. dim. } 2\}$ modulo diffeomorphism.

   We want a *geometric compactification* $\overline{\mathscr{M}}$ of $\mathscr{M}$, i.e. a compactification with respect to some natural distance on $\mathscr{M}$. A natural distance is the *Gromov-Hausdorff distance* which is a fascinating definition.

   This has been somewhat achieved in dimension 2 (see [10, 11]), but in higher dimensions there are many open questions. Also, there are many interesting moduli spaces apart from Calabi-Yau manifolds.

References

[1] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978), no. 1711, 425–461. MR 506229 38

[2] Arthur L. Besse, *Einstein manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008, Reprint of the 1987 edition. MR 2371700 56

[3] Xiuxiong Chen, Simon Donaldson, and Song Sun, *Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities*, J. Amer. Math. Soc. **28** (2015), no. 1, 183–197. MR 3264766 65

[4] Hannah de Lázari, Jason D. Lotay, Henrique Sá Earp, and Eirik Eik Svanes, *Local descriptions of the heterotic su(3) moduli space*, 2024. 64

[5] Simon K Donaldson, *Some numerical results in complex differential geometry*, arXiv preprint math/0512625 (2005). 65

[6] Hans-Joachim Hein and Song Sun, *Calabi-yau manifolds with isolated conical singularities*, Publications mathématiques de l'IHÉS **126** (2017), no. 1, 73–130. 64, 65

[7] Daniel Huybrechts, *Complex geometry*, Universitext, Springer-Verlag, Berlin, 2005, An introduction. MR 2093043 5, 7, 16, 24

[8] Dominic D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. MR 1787733 64

[9] Yang Li, *Survey on the metric syz conjecture and non-archimedean geometry*, International Journal of Modern Physics A **37** (2022), no. 17, 2230009. 65

[10] Yuji Odaka and Yoshiki Oshima, *Collapsing k3 surfaces and moduli compactification*, (2018). 65

[11] Song Sun and Ruobing Zhang, *Collapsing geometry of hyperk\" ahler 4-manifolds and applications*, arXiv preprint arXiv:2108.12991 (2021). 65

[12] Song Sun and Ruobing Zhang, *Complex structure degenerations and collapsing of calabi-yau metrics*, 2024. 65

[13] Gang Tian and Shing Tung Yau, *Complete kähler manifolds with zero ricci curvature ii*, Inventiones mathematicae **106** (1991), 27–60. 64