An application of numerical techniques to rigorous proof in special holonomy

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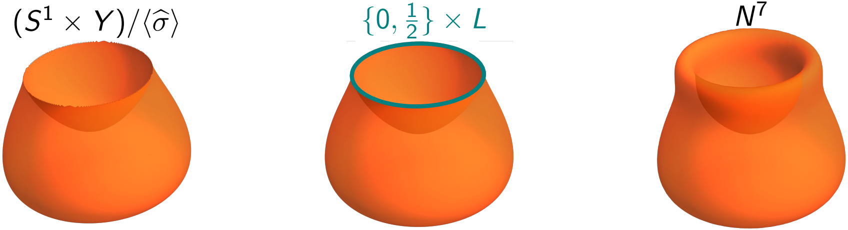
Abstract: Approximations of Calabi-Yau metric are a popular tool to produce heuristics, but so far have not been leveraged to rigorously prove theorems in geometry. I present one work in progress, in which we prove that the real loci of certain Calabi-Yau manifolds admit harmonic nowhere vanishing 1-forms, which are needed for an application in G2-geometry. I will explain the proof strategy, which consists of two parts: first, I formulate an estimate for the difference between approximate metric and true Calabi-Yau metric in terms of the Ricci curvature of the approximate metric which is of independent interest. Second, I explain the connection between nowhere vanishing 1-forms with respect to the two different metrics. This is joint work with Rodrigo Barbosa, Michael Douglas, and Yidi Qi.

Pure maths and machine learning

- Use machine learning for conjecture generation
  e.g. [Davies et al., 2021]: conjecture connecting algebraic and geometric properties of knots
- Machine learning applied to pure mathematics datasets
  e.g. [He, 2017]: inputs are Calabi-Yau manifolds, outputs are their Hodge numbers (previously computed exactly)
- Numerical verification methods for PDE solutions
  e.g. [Nakao et al., 2019]: proof there exists smooth solution near a finite element solution to Navier-Stokes equation

Background

- Let Y be Calabi-Yau 3-fold with Calabi-Yau metric g_{CY}
- $\sigma : Y \to Y$ anti-holomorphic involution, $L := fix(\sigma)$
  example: quintic with real coefficients in $\mathbb{CP}^4$ and $\sigma([z_0 : \cdots : z_4]) = [\overline{z_0} : \cdots : \overline{z_4}]$
- $S^1 \times Y$ has dimension 7 and holonomy $SU(3)$. Problem: want holonomy G_2
- Define $\widehat{\sigma} : S^1 \times Y \to S^1 \times Y$ as $(x, y) \mapsto (-x, \sigma(y))$



Theorem ([Joyce and Karigiannis, 2017])

If there exists $\lambda \in \Omega^1(L)$ harmonic w.r.t. $g_{CY}|_L$ that is nowhere 0, then there exists a resolution $N^7 \to (S^1 \times Y)/\langle\widehat{\sigma}\rangle$ with holonomy equal to G_2.

- Goal: check if such a 1-form exists
- First Betti number $\to$ harmonic 1-forms. Nowhere 0? Must know the metric!

Strategy for checking if nowhere zero harmonic $\lambda \in \Omega^1(L)$ exists

Goal: Check if L admits harmonic nowhere zero 1-form

Step 1: Approximate g_{CY} by g_{approx}

Step 2: Prove: for all $\epsilon_1 > 0$ exists $\delta_1 > 0$ such that:

$$\text{if } ||Ric(g_{approx})||_{C^0} < \delta_1, \text{ then } ||g_{CY} - g_{approx}||_{C^1} < \epsilon_1.$$

Step 3: Find $\lambda \in \Omega^1(L)$ harmonic w.r.t. g_{approx} and compute $\min_{x \in L} |\lambda(x)|$.

Step 4: Prove: for all $\epsilon_2 > 0$ exists $\delta_2 > 0$ such that:

if $\lambda \in \Omega^1(L)$ harmonic w.r.t g_{approx} and $||\lambda||_{L^2} = 1$ and $\min_{x \in L} |\lambda(x)| > \epsilon_2$ and $||g_{CY} - g_{approx}||_{C^1} < \delta_2$, then exists $\eta \in \Omega^0(L)$ s.t. $\lambda + d\eta$ is nowhere 0 and harmonic w.r.t. g_{CY}.

Result:

- Compute g_{approx}
- Check that $||Ric(g_{approx})||_{C^0}$ is small
- Find $\lambda \in \Omega^1(L)$ harmonic w.r.t. g_{approx} s.t. $\min |\lambda|$ is big
- Then exists $\eta \in \Omega^0(L)$ s.t. $\lambda + d\eta$ harmonic w.r.t. g_{CY} and nowhere 0

Step 1: approximate g_{CY} by g_{approx}

- Holomorphic volume form locally $\Omega = dz^1 \wedge dz^2 \wedge dz^3 \rightsquigarrow \mathrm{vol}_\Omega := \Omega \wedge \overline{\Omega} \in \Omega^6(Y)$
- Ample line bundle $L \to Y$ and $l \in \mathbb{N}$ such that $L^{\otimes l}$ very ample
  Example: $Y \subset \mathbb{CP}^4$ quintic, $(O(1)|_Y)^{\otimes l}$
- $s_1, \ldots, s_N \in H^0(L^{\otimes l})$ basis of holomorphic sections
  $\Rightarrow$ embedding $s = (s_1, \ldots, s_N) : Y \to \mathbb{CP}^{N-1}$
- h positive definite Hermitian form on $H^0(L^{\otimes l}) \rightsquigarrow$ Fubini-Study metric
  Kähler potential: $\log \sum_{i,j} h_{i,j} s^i \bar{s}^j$. Volume form: $\mathrm{vol}_h \in \Omega^6(Y)$
- [Donaldson, 2009]: choose h cleverly to approximate CY metric
  (Ignoring a constant) If $\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} = 1$, then Ricci-flat
- Choose $x_1, \ldots, x_k \in Y$ and find

$$\min_h \int_{\{x_1,\ldots,x_k\}} \left(\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} - 1 \right)^2$$

- Convenient: there is fast machine learning software to find local minima
  [Douglas et al., 2022]

Step 2: $||Ric(g_{approx})||$ small $\Rightarrow$ $||g_{CY} - g_{approx}||$ small

Theorem (Yau's theorem. [Yau, 1978] and p.105-107 in [Joyce, 2000])

(Y, ω, g) *Kähler manifold of cx. dimension* m *with holomorphic volume form* $\Omega \in \Omega^m(Y, \mathbb{C})$. *Then there exists a unique* $K \in \Omega^0_{mean\text{-}0}(Y)$ *s.t.*

i. $\omega + \mathrm{dd}^c K$ *is Kähler*

ii. $(\omega + \mathrm{dd}^c K)^m = \Omega \wedge \overline{\Omega}$ *(ignoring a constant)*

iii. *(New) Up to cx. dimension 3: there exists* C *depending on* $||f-1||_{C^0}$ *and* $||f^{-1} - 1||_{C^0}$ *such that* $||\mathrm{dd}^c K||_{C^1} < C$, *where* $f = \frac{\mathrm{vol}_\Omega}{\mathrm{vol}_\omega}$ *and* $C \to 0$ *as* $f \to 1$

Proof of iii.: Yau proves estimate $||\mathrm{dd}^c K||_{C^0} \leq C$ for some C, now want C small.
Computations around $p \in Y$ from Yau's proof: ex. functions a_1, a_2, a_3 s.t.

$$\Delta K = 3 - \sum a_j \leq 3 \cdot (1 - f), \qquad -\Delta K = 3 - \sum a_j^{-1} \leq 3 \cdot \left(1 - f^{-1}\right),$$

$$\prod a_j = f(p), \qquad |\mathrm{dd}^c K|^2 = 2 \sum (a_j - 1)^2.$$

So $f \approx 1 \Rightarrow \Delta K \approx 0$ (first two eqns) $\Rightarrow a_1 \approx a_2 \approx a_3 \approx 1$ (first three equations, only in dim ≤ 3) $\Rightarrow \mathrm{dd}^c K \approx 0$ (last eqn) □

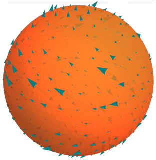
$\rightsquigarrow$ Then expect C^k-estimates, because K satisfies elliptic equation

Step 3: Find $\lambda \in \Omega^1(L)$ harmonic w.r.t. g_{approx}

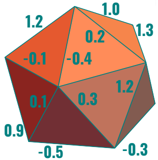
- $\mathcal{T}$ simplicial complex, triangulation of L
- Discrete exterior calculus [Hirani, 2003]: k-forms $\Omega^k(\mathcal{T}) := \mathrm{Hom}\left(\bigoplus_{\sigma \in \mathcal{T}^k} \mathbb{R}\sigma, \mathbb{R}\right)$

  Discretisation $R : \Omega^k(L) \to \Omega^k(\mathcal{T})$

  $$\omega \mapsto \left(\sigma \mapsto \int_\sigma \omega\right)$$



$\mapsto$



  Have $d_{\mathcal{T}}$, $d^*_{\mathcal{T}}$, $\Delta_{\mathcal{T}}$ on $\mathcal{T}$

- Fix $\omega \in \Omega^1(L)$ closed $\overset{\text{Hodge thm}}{\Longrightarrow}$ unique $\eta \in \Omega^0_{\text{mean-0}}(L)$ s.t. $\omega + d\eta$ harmonic
- $\overset{\text{discrete Hodge thm}}{\Longrightarrow}$ unique $\eta_{\mathcal{T}} \in \Omega^0_{\text{mean-0}}(\mathcal{T})$ s.t. $R\omega + d_{\mathcal{T}}\eta_{\mathcal{T}}$ is $\Delta_{\mathcal{T}}$-harmonic

Conjecture ([Schulz and Tsogtgerel, 2020])

$||R\eta - \eta_{\mathcal{T}}||_{C^1_{\mathcal{T}}} = \mathcal{O}(\mathrm{diam}(\mathcal{T})^2)$

- Remark: in FEM get L^2-estimates; on space $\Omega^k(\mathcal{T})$ all norms equivalent $\rightsquigarrow C^1$
- $\Rightarrow$ if $R\omega + d_{\mathcal{T}}\eta_{\mathcal{T}}$ far away from 0, then $\omega + d\eta$ nowhere 0

Step 4: Perturb g_{approx}-harmonic to g_{CY}-harmonic

Theorem

For all $\epsilon_2 > 0$ exists $\delta_2 > 0$ such that:

if $\lambda \in \Omega^1(L)$ harmonic w.r.t g_{approx} and $||\lambda||_{L^2} = 1$ and $\min_{x \in L} |\lambda(x)| > \epsilon_2$ and $||g_{CY} - g_{approx}||_{C^1} < \delta_2$, then exists $\eta \in \Omega^0(L)$ s.t. $\lambda + \mathrm{d}\eta$ is nowhere 0 and harmonic w.r.t. g_{CY}.

Proof: $\Delta_{approx}(\lambda) = 0 \Rightarrow ||\Delta_{CY}(\lambda)||_{L^2} \leq C \cdot \delta_2$
Let $\eta \in \Omega^0_{0\text{-mean}}(L)$ s.t. $\Delta_{CY}(\eta) = -\mathrm{d}^*\lambda \Rightarrow \Delta_{CY}(\lambda + \mathrm{d}\eta) = 0$

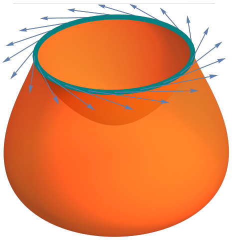
$$||\eta||_{C^{0,\alpha}} \lesssim ||\eta||_{L^2_2} \lesssim ||\Delta_{CY}\eta||_{L^2} = ||\mathrm{d}^*_{CY}\lambda||_{L^2} \lesssim ||\Delta_{CY}\lambda||_{L^2} \lesssim \delta$$

by Sobolev embedding and elliptic regularity (to do: C^1-estimate)
Then $\min|\lambda + \mathrm{d}\eta| \geq (\min|\lambda|) - (\max \mathrm{d}\eta) \geq \epsilon_2 - C \cdot \delta_2$. Bigger than 0 for δ_2 small □

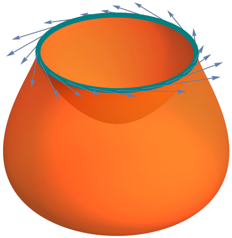
- Need all constants explicit!
- For $||\eta||_{L^2_3} \lesssim ||\Delta_{CY}\eta||_{L^2_1}$ need smallest eigenvalue of Δ, e.g. [Li and Yau, 1980]

Result

1. Compute g_{approx} and compute $\left|\left|\frac{\mathrm{vol}_\Omega}{\mathrm{vol}_\omega} - 1\right|\right|_{L^\infty}$ and $\left|\left|\frac{\mathrm{vol}_\omega}{\mathrm{vol}_\Omega} - 1\right|\right|_{L^\infty}$, hopefully small
2. $\overset{\text{step 2}}{\Longrightarrow}$ $||g_{CY} - g_{approx}||_{C^1}$ small
3. Find $\lambda \in \Omega^1(L)$ harmonic w.r.t. g_{approx}, hopefully large bound from below
4. $\overset{\text{step 4}}{\Longrightarrow}$ exists nowhere 0 harmonic 1-form w.r.t. g_{CY}



Harmonic 1-form w.r.t. g_{approx}



Perturbed 1-form, it has a worse bound from below w.r.t.

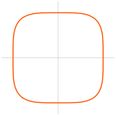
g_{CY}, but still nowhere 0

Example 0: [Joyce and Karigiannis, 2017, Example 7.6]

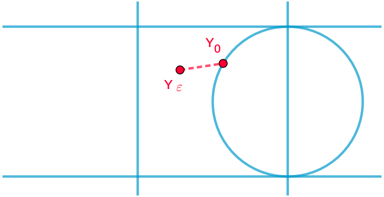
- Singular Calabi-Yau
  $Y_0 := \{([w_0 : w_1], [x_0 : x_1], [y_0 : y_1], [z_0 : z_1]) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 :$
  $(w_0 x_0 y_0 z_0)^2 + (w_1 x_1 y_1 z_1)^2 = 0\}$
- Real locus $Y_0(\mathbb{R}) \equiv T^3$ smooth
  $\Rightarrow$ small perturbation Y is smooth, still has $Y(\mathbb{R}) = T^3$
- Conjecture: metric on T^3 is close to flat metric. $\Rightarrow$ exist nowhere zero 1-forms
- Proof idea (by Yang Li):
  - Y_0 admits singular Calabi-Yau metric g_0 [Eyssidieux et al., 2009]]
  - Y_0 has complex T^3 symmetry $\Rightarrow$ isometric T^3-action w.r.t. g_0
  - Let Y_ϵ be a 1-parameter family of Calabi-Yaus with $Y_{\epsilon=0} = Y_0$ and metric g_ϵ
  - Then $g_\epsilon \to g_0$ away from singularities of Y_0 [Rong and Zhang, 2011]
  - $\Rightarrow g_\epsilon|_{T^3}$ approximately flat

Example 1: $S^1 \times S^2$

$$Q = x_1^4 + x_2^4 - 1 \qquad S = x_3^2 + x_4^2 + x_5^2 - 1$$



- $Y_0^{aff} := Z(Q) \cap Z(S) \subset \mathbb{C}^5$. Viewed projective is $Y_0 \subset \mathbb{CP}^4$
- Y_0 singular, real locus $Y_0(\mathbb{R}) \simeq S^1 \times S^2$ smooth
- Small perturbation Y_ϵ smooth and has $Y_\epsilon(\mathbb{R}) \simeq S^1 \times S^2$, may be an example
- Problem: for epsilon small, have $\frac{\mathrm{vol}_\Omega}{\mathrm{vol}_\omega}$ of g_{approx} large $\rightsquigarrow$ programme fails at step 4
- Potential solution: find perturbation of Y_0 with maximum distance to singular Calabi-Yaus (suggested in [Douglas et al., 2022] for quintics)



- Check topological type of Y_ϵ using persistent homology [Di Rocco et al., 2022]

Example 2: from real cubics

- Diffeomorphism type of real cubics $C \subset \mathbb{RP}^4$ classified in [Krasnov, 2009]: Possible are $\mathbb{RP}^3 \# (S^1 \times S^2) \# \dots \# (S^1 \times S^2)$ with $0, 1, 2, 3, 4, 5$ handles (plus $\mathbb{RP}^3 \cup S^3$ and one exotic possibility that is not understood)
- Then $Q = Z(C \cdot (x_0^2 + \dots + x_4^2))$ quintic
- Smooth in $\mathbb{RP}^4$, perturb to be smooth in $\mathbb{CP}^4 \rightsquigarrow Y_\epsilon$
- If C has harmonic nowhere zero 1-form $\Rightarrow$ closed nowhere zero 1-form $\Rightarrow$ Tischler's theorem: C is a fibration over S^1
- Topological condition, not satisfied for these diffeomorphism types
- In that case: $\lambda \in \Omega^1(C)$ must have zeros; use steps 1-4 to check how many
- Conjecture: resolution construction for 1-forms with zeros yields orbifolds with isolated conical singularities; local analysis around zeros not yet worked out

Thank you for the attention!

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