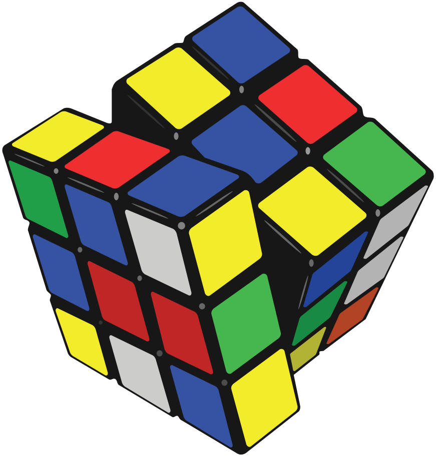
What is a *group* and why should I care?

Daniel Platt

October 10, 2019

What is a Group?

Very general mathematical concept, can be applied to:



Rubik's Cube

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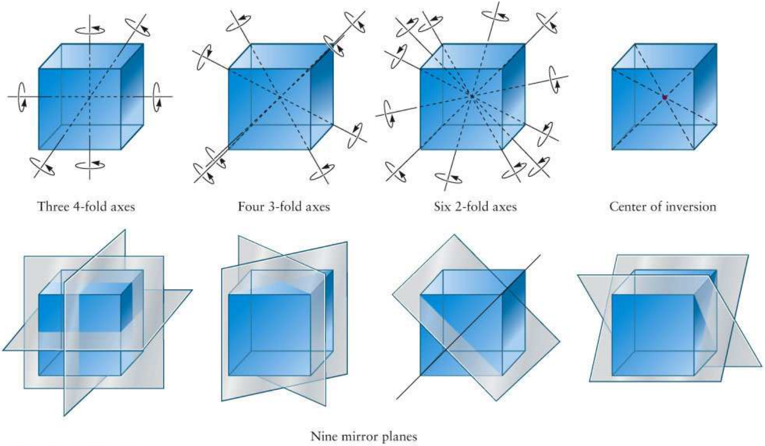

Symmetry Group of the Cube

What is a Group?

Very general mathematical concept, can be applied to:

$$\mathbb{R}$$

The Real Numbers

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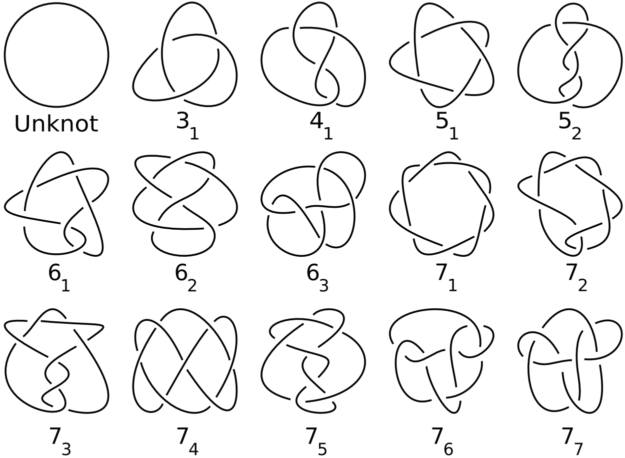

Knot Groups

What is a Group?

Real life applications:

$$\text{https}\underline{\text{s}}\text{://www...}$$

"Elliptic Curves Cryptography": send messages across the internet that can only be read by the recipient

What is a Group?

Real life applications:

Infrared Spectroscopy: Find out what molecules are contained in a sample without having to touch it

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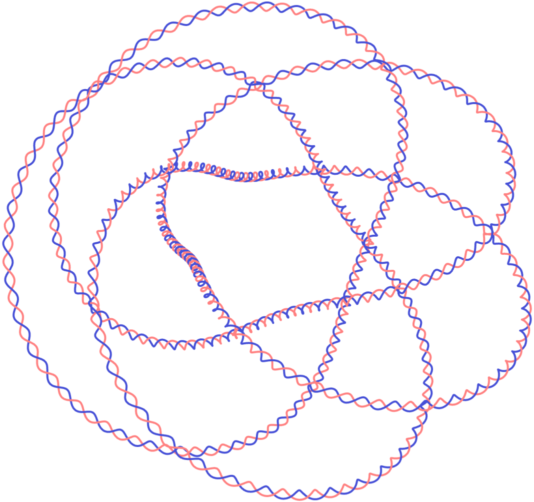
Real life applications:



DNA and braid groups: DNA is a long thing, tangled up; biologists want to understand how exactly it is tangled

Mathematical Definition

Definition

A *group* is a set of elements together with an operation that combines any two elements to form a third element, satisfying some properties.

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Set $= \{\spadesuit, \clubsuit, \heartsuit\}$, operation $\circ$ given by

$\circ$	$\spadesuit$	$\clubsuit$	$\heartsuit$
$\spadesuit$	$\spadesuit$	$\clubsuit$	$\heartsuit$
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E.g. $\spadesuit \circ \heartsuit = \heartsuit$,

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$\circ$	♠	♣	♡
♠	♠	♣	♡
♣	♣	♡	♠
♡	♡	♠	♣

Neutral element:
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Picture Hanging Puzzles

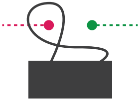
Task: Hang a picture on two nails, so that it falls down if *either* nail is pulled out.

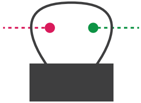
Picture Hanging Puzzles

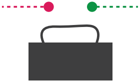
Idea: Write path of the rope as formula

If rope passes left nail write a if it crosses the dotted line clockwise and a^{-1} for counter-clockwise

Analog for right nail with letters b and b^{-1}







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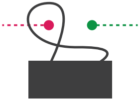

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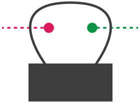
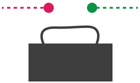


ab



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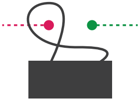

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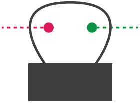

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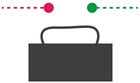

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a^{-1} $\qquad$ ab $\qquad$ 0 $\qquad$ aba^{-1}

Group Structure

for rope formulae: write next to each other

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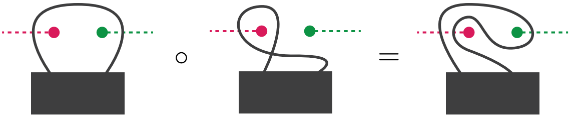

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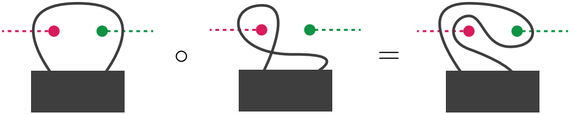

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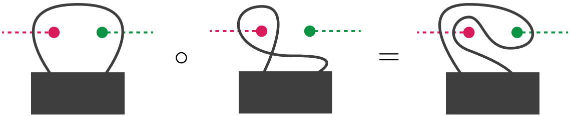

1. What is the neutral element here?
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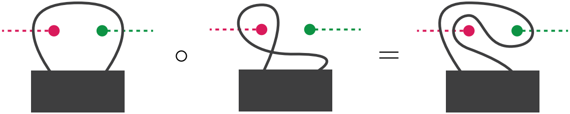

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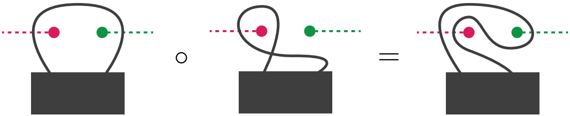

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Solution for the puzzle? How about 3 nails?

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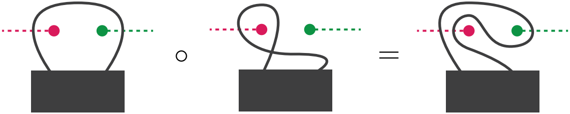

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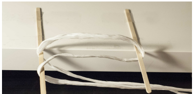

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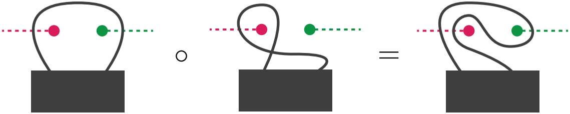

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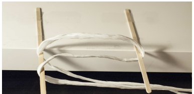

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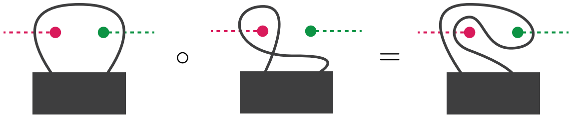

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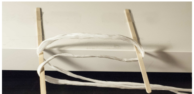

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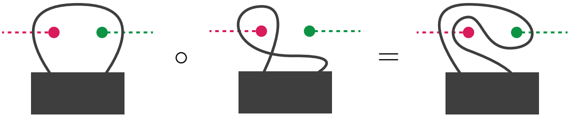

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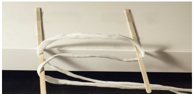

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$$aba^{-1}b^{-1}$$

$$\text{pull out left nail } \rightarrow \cancel{a}b\cancel{a^{-1}}b^{-1} = bb^{-1} = 0$$

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$$xcx^{-1}c^{-1} = aba^{-1}b^{-1}cbab^{-1}a^{-1}c^{-1}$$

3. What about n nails?

Solution

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& aba^{-1}b^{-1} \\
\text{pull out left nail } & \to \cancel{a}b\cancel{a^{-1}}b^{-1} = bb^{-1} = 0 \\
\text{pull out right nail } & \to a\cancel{b}a^{-1}\cancel{b^{-1}} = aa^{-1} = 0
\end{aligned}
$$

For three nails: write $x = aba^{-1}b^{-1}$

1. What is x^{-1}? (I.e. x "grouped with" what gives 0?)
   $x^{-1} = bab^{-1}a^{-1}$
2. How can you construct a solution from c, c^{-1}, x, x^{-1}?

$$xcx^{-1}c^{-1} = aba^{-1}b^{-1}cbab^{-1}a^{-1}c^{-1}$$

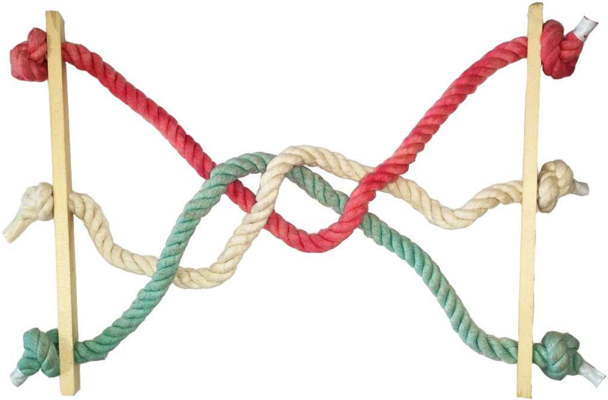
3. What about n nails?
   Let y be a solution for $n-1$ nails, then $yny^{-1}n^{-1}$ is a solution for n nails

Braid Groups

2 sticks with 3 parallel strings

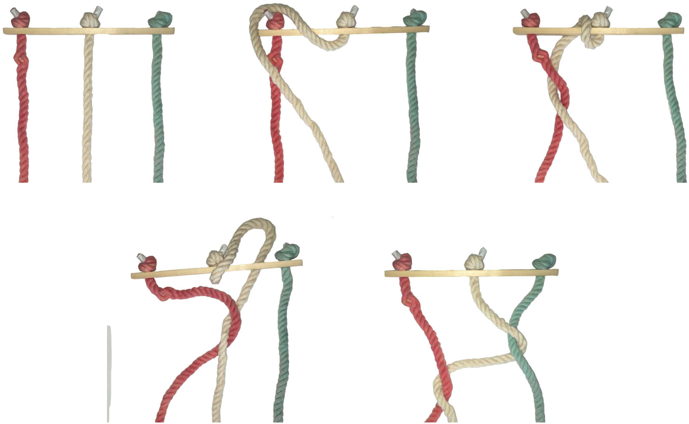
rotate bottom stick by 360°

Question: you are allowed to rotate the strings around the sticks. Can the strings be untangled? What about a 720° rotation?



Braid Groups

"rotating a string around a stick" means to take the string all the way around:

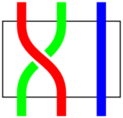


(In the above example: Don't stop after pulling the white string half way around the stick, i.e. after crossing the white and the red string. If that was allowed, the puzzle would be too easy)

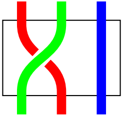
Braid Groups

Idea: Write braid as *braid word*

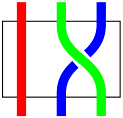
$$\text{left strand } \textit{over} \text{ middle strand} = s_1, \quad \text{left } \textit{under} \text{ middle} = s_1^{-1}$$
$$\text{middle strand } \textit{over} \text{ right strand} = s_2, \quad \text{middle } \textit{under} \text{ right} = s_2^{-1}$$



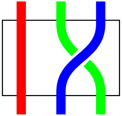
s_1



s_1^{-1}



s_2



s_2^{-1}

Group Structure

for braid words:

Group Structure

for braid words: write words next to each others

Group Structure

for braid words: write words next to each others

$$(s_1 s_1) \circ (s_2^{-1}) = s_1 s_1 s_2^{-1}$$

Group Structure

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$$(s_1 s_1) \circ (s_2^{-1}) = s_1 s_1 s_2^{-1}$$

for pictures: arrange under each other

Group Structure

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$$(s_1 s_1) \circ (s_2^{-1}) = s_1 s_1 s_2^{-1}$$

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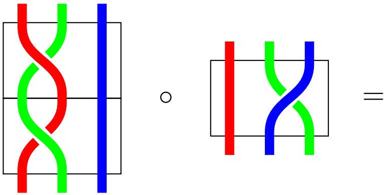
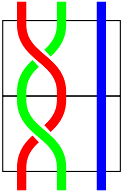


Group Structure

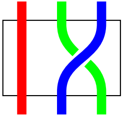
for braid words: write words next to each others

$$(s_1 s_1) \circ (s_2^{-1}) = s_1 s_1 s_2^{-1}$$

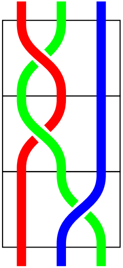
for pictures: arrange under each other



$\circ$



$=$

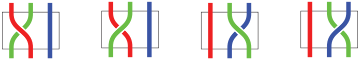


Finding Braid Words for Braids

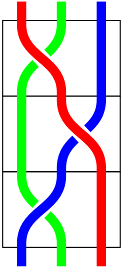
left strand over middle strand = s_1, left under middle = s_1^{-1}

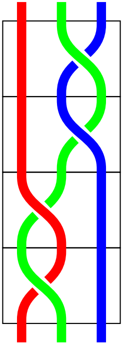
middle strand over right strand = s_2, middle under right = s_2^{-1}

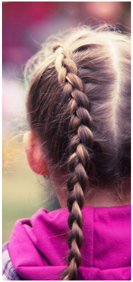
What are the braid words for these braids?

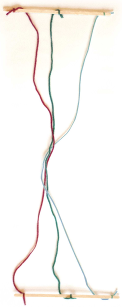


s_1 s_1^{-1} s_2 s_2^{-1}









Finding Braid Words for Braids

What are the braid words for these braids?

left strand over middle strand = s_1, left under middle = s_1^{-1}
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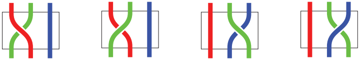
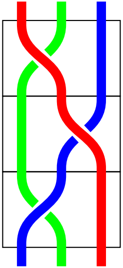
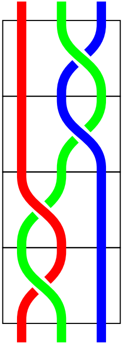
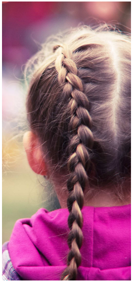


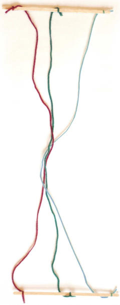
s_1 s_1^{-1} s_2 s_2^{-1}



$s_1 s_2 s_1^{-1}$







Finding Braid Words for Braids

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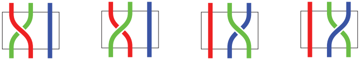
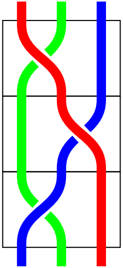
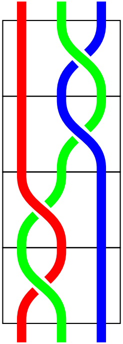
What are the braid words for these braids?



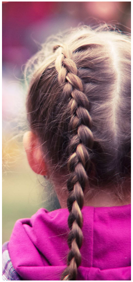
s_1 s_1^{-1} s_2 s_2^{-1}

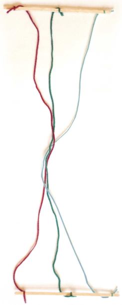


$s_1 s_2 s_1^{-1}$



$s_2 s_2 s_1 s_1$





Finding Braid Words for Braids

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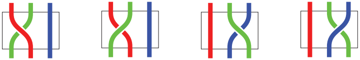
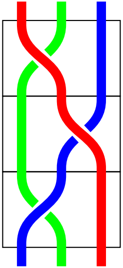
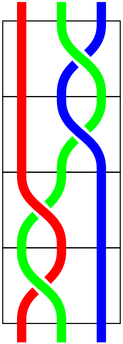


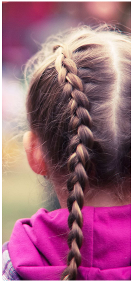
s_1 s_1^{-1} s_2 s_2^{-1}

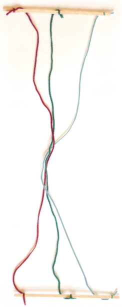


$s_1 s_2 s_1^{-1}$



$s_2 s_2 s_1 s_1$





$s_1^{-1} s_2 s_1^{-1} s_2 s_1^{-1} s_2 \dots$

Finding Braid Words for Braids

What are the braid words for these braids?

left strand over middle strand = s_1, left under middle = s_1^{-1}

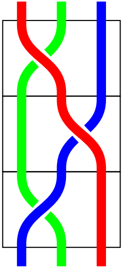
middle strand over right strand = s_2, middle under right = s_2^{-1}

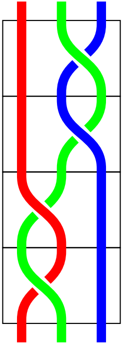

s_1

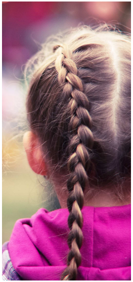

s_1^{-1}

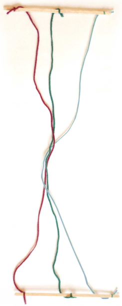

s_2


s_2^{-1}


$s_1 s_2 s_1^{-1}$


$s_2 s_2 s_1 s_1$

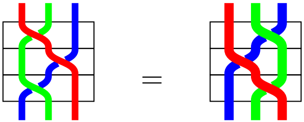

$s_1^{-1} s_2 s_1^{-1} s_2 s_1^{-1} s_2 \ldots$


$s_1 s_2 s_1 s_2 s_1 s_2$

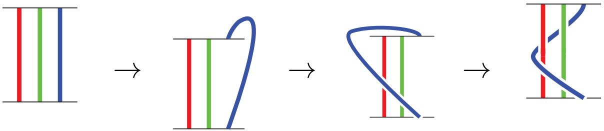
s_1 $\quad$ s_1^{-1} $\quad$ s_2 $\quad$ s_2^{-1}

Solving the puzzle: How does the braid word change when

1. rearranging crossings?



2. moving one of the outermost strings around one stick?



Use 1. and 2. to go from the formula for the 360° braid or 720° braid to the trivial formula. Is it possible?

Try something like: $s_1\underline{s_2 s_1 s_2}s_1 s_2 = s_1 s_1\underline{s_2 s_1 s_1 s_2} = s_1 s_1$

s_1 s_1^{-1} s_2 s_2^{-1}

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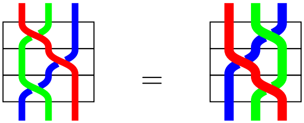
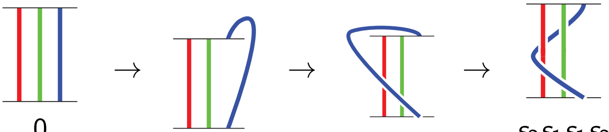


$s_1 s_2 s_1$ $s_2 s_1 s_2$

2. moving one of the outermost strings around one stick?



0 $s_2 s_1 s_1 s_2$

Use 1. and 2. to go from the formula for the 360° braid or 720° braid to the trivial formula. Is it possible?

Try something like: $s_1 \underline{s_2 s_1 s_2} s_1 s_2 = s_1 s_1 \underline{s_2 s_1 s_1 s_2} = s_1 s_1$

1. What are the braid words of the 360° and 720° braids?
2. How can we change the braid word with our allowed motions?
3. Can one apply the rules from 2. and 3. to go from the braid word of the 720° braids to the neutral element?

"Take every string around the stick exactly once!"

4. What about the 360° braid?

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   360° : $s_1 s_2 s_1 s_2 s_1 s_2$, 720° : $s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2$
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2. How can we change the braid word with our allowed motions?
   $s_1 s_2 s_2 s_1 = 0$, $s_2 s_1 s_1 s_2 = 0$, $s_2 s_2 s_1 s_1 = 0$, $s_1 s_2 s_1 = s_2 s_1 s_2$
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   $360^\circ: s_1 s_2 s_1 s_2 s_1 s_2$, $720^\circ: s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2$
2. How can we change the braid word with our allowed motions?
   $s_1 s_2 s_2 s_1 = 0$, $s_2 s_1 s_1 s_2 = 0$, $s_2 s_2 s_1 s_1 = 0$, $s_1 s_2 s_1 = s_2 s_1 s_2$
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$$
\begin{aligned}
(s_1 s_2)^6 &= s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 \\
&= s_1 s_2 s_2 s_1 s_2 s_2 s_1 s_1 s_2 s_1 s_1 s_2 \\
&= \underbrace{s_1 s_2 s_2 s_1}\,\underbrace{s_2 s_2 s_1 s_1}\,\underbrace{s_2 s_1 s_1 s_2} = 0
\end{aligned}
$$

"Take every string around the stick exactly once!"

4. What about the 360° braid?

1. What are the braid words of the 360° and 720° braids?
   360° : $s_1 s_2 s_1 s_2 s_1 s_2$, 720° : $s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2$
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   $s_1 s_2 s_2 s_1 = 0$, $s_2 s_1 s_1 s_2 = 0$, $s_2 s_2 s_1 s_1 = 0$, $s_1 s_2 s_1 = s_2 s_1 s_2$
3. Can one apply the rules from 2. and 3. to go from the braid word of the 720° braids to the neutral element?

$$\begin{aligned}
(s_1 s_2)^6 &= s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 \\
&= s_1 s_2 s_2 s_1 s_2 s_2 s_1 s_1 s_2 s_1 s_1 s_2 \\
&= \underbrace{s_1 s_2 s_2 s_1}\,\underbrace{s_2 s_2 s_1 s_1}\,\underbrace{s_2 s_1 s_1 s_2} = 0
\end{aligned}$$

"Take every string around the stick exactly once!"

4. What about the 360° braid? Impossible! $s_1 s_2 s_1 s_2 s_1 s_2$ has 6 letters. Applying rules changes number by 4 $\rightarrow$ can never reach 0