From Zero to Riemannian Manifolds in 8 Pages

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24th November 2023

Contents

1 Introduction

The purpose of this document is to explain what a Riemannian metric on an affine variety is, and to briefly mention Calabi-Yau metrics without rigorously defining them. Roughly speaking: varieties are curved spaces, for example the sphere. A Riemannian metric is a way to measure distances on this space. And a Calabi-Yau metric is a special kind of Riemannian metric.

In Section 2 we explain how to measure vectors in *vector spaces*, not yet on curved spaces. This is done by real inner products and complex inner products. In Section 3 we define the first type of curved spaces, called *affine varities*. However, complex affine varieties are never compact, but we are interested in compact Calabi-Yau manifolds. Therefore, in Section 4 we define a second type of curved spaces, called *projective varieties*. They are always compact. In Section 5 we define Riemannian metrics, which really are just a real inner product in every point of a variety. There are technical difficulties in defining these on projective varieties, so we only define them on affine varieties. Last, in Section 6 we write a bit about Calabi-Yau manifolds, but will not define them rigorously.

Throughout, we try to always give two references for everything: one book and one freely available online resource.

2 Linear Algebra

In this section we will define inner product spaces, namely in Definitions 2.1 and 2.2. Inner products are extra structures that can be put on a vector space. The Euclidean space $\mathbb{R}^n$ is a special case of a vector space, and it comes with a standard inner product, namely the dot product. For general vector spaces, there is no standard inner product, but one can still define what an inner product is, which is a prerequisite to define Riemannian metrics later on. References are: [Strang, 2022, Section 2.7], [Dawkins, 2023, Section Inner Product Spaces].

We begin with the definition of inner product spaces:

Definition 2.1. Let V be a real vector space. A real inner product is a map $B : V \times V \to \mathbb{R}$ satisfying:

- for all $x, y \in V$ we have $B(x, y) = B(y, x)$ ("symmetric"),
- for all $\lambda, \mu \in \mathbb{R}$ and $x, y \in V$ we have $B(\lambda x, \mu y) = \lambda\mu B(x, y)$ ("bilinear"),
- for all $x \in V$ with $x \neq 0$ we have $B(x, x) > 0$ ("positive definite").

We denote the set of all real inner products on V by $\mathrm{Bil}(V)$.

Definition 2.2. Let W be a complex vector space. A complex inner product is a map $B : W \times W \to \mathbb{C}$ satisfying:

- for all $x, y \in W$ we have $B(x, y) = \overline{B(y, x)}$ ("symmetric"),
- for all $\lambda, \mu \in \mathbb{C}$ and $x, y \in W$ we have $B(\lambda x, \mu y) = \lambda\overline{\mu}B(x, y)$ ("bilinear"),
- for all $x \in W$ with $x \neq 0$ we have $B(x, x) \in \mathbb{R}$ and $B(x, x) > 0$ ("positive definite").

Here, $\overline{(\cdot)}$ denotes the complex conjugate of a complex number. We denote the set of all complex inner products on W by $\mathrm{Bil}(W)$.

The Euclidean space $\mathbb{R}^n$ is a special case of a vector space, and it has a standard inner product, as explained in the following example. The corresponding complex vector space $\mathbb{C}^n$ also has a similar standard inner product, which is the second example:

Example 2.3. The map

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R} \\ (x, y) &\mapsto x^T y \end{aligned}$$

is a *real inner product*. It is called *dot product* or Euclidean inner product. The map

$$\begin{aligned} \mathbb{C}^n \times \mathbb{C}^n &\to \mathbb{C} \\ (x, y) &\mapsto x^* y \end{aligned}$$

is a *complex inner product*. Here, $(\cdot)^T$ denotes the transpose of a column vector, and $(\cdot)^*$ denotes the conjugate transpose of a column vector.

On $\mathbb{R}^n$ and $\mathbb{C}^n$ inner products are actually in one-to-one correspondence with certain matrices. Here is the definition of these matrices:

Definition 2.4. A matrix $A \in \mathbb{R}^{n\times n}$ is called

- *symmetric* if $A^T = A$,
- *positive definite* if $x^T Ax > 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$.

A matrix $A \in \mathbb{C}^{n\times n}$ is called

- *hermitian* if $A^* = A$,
- *positive definite* if $x^* Ax > 0$ for all $x \in \mathbb{C}^n$ with $x \neq 0$.

And here is the Lemma stating how to go from matrices to inner products:

Lemma 2.5. *Let $A \in \mathbb{R}^{n\times n}$. Define the following map:*

$$\begin{aligned} B_A : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R} \\ (x, y) &\mapsto x^T Ay. \end{aligned}$$

Then B_A is a real inner product if and only if A is symmetric and positive definite.

An analog result of this lemma holds for complex inner products and hermitian matrices. For an abstract vector space V, one can choose a basis, which defines an isomorphism $\phi : V \longrightarrow \mathbb{R}^n$ if the vector space is real, and an isomorphism with $\phi : V \longrightarrow \mathbb{C}^n$ if the vector space is complex. Either way, if B is an inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$, then $B(\phi(v), \phi(w))$ for $v, w \in V$ is an inner product on V. So, also in this abstract case, one gets inner products on V from symmetric or hermitian matrices.

Example 2.6. If one defines $A := \mathrm{Id} := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$ to be the identity matrix, then the map B_A from Lemma 2.5 is exactly the standard inner product from Example 2.3. (This is true for $\mathbb{R}^n$ as well as $\mathbb{C}^n$.)

3 Affine Varieties

Up until now, all definitions were made for *vector spaces*. Vector spaces have the property that the sum of two vectors is again a vector. We will now define *varieties*, which are a certain type of curved space. Importantly, for curved spaces it is not true that the sum of two points on the variety is again a point on the variety. Varieties are the basic objects in the huge field of *algebraic geometry*. The most important object in this section is the *tangent space* of a variety from Definition 3.5. References are: [Gathmann, 2021, Chapter 1], [Holme, 2011, Section I.1], which contain much more complicated information about algebraic geometry than presented here. The reader should be warned that the definition of tangent space made below is a characterisation from *differential geometry*, not algebraic geometry. It can be found in [O'neill, 2006, Lemma 3.8].

We begin with the definition of affine varieties:

Definition 3.1. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial. The set

$$Z(f) := \{(x_1, \dots, x_n) \in \mathbb{C}^n : f(x_1, \dots, x_n) = 0\}$$

is called the *zero locus of* f or *vanishing locus of* f. It is called an *affine variety*.

It is sometimes denoted by $V(f)$ instead of $Z(f)$. Some shapes are smooth, some shapes are spiky, and this is formalised in the following definition:

Definition 3.2. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial and write $M := Z(f)$. The set

$$\mathrm{sing}(M) := \{x \in M : (\nabla f)(x) = 0\}$$

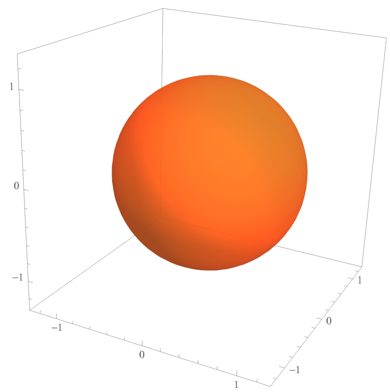
is called *singular set of* M, and $\mathrm{reg}(M) := M \setminus \mathrm{sing}(M)$ is called *regular set of* M.

Here are two example varieties:

Example 3.3. Let $f = x_1^2 + x_2^2 + x_3^2 - 1$. Then $M := Z(f)$ is a subset of $\mathbb{C}^3$. We cannot draw so many complex numbers, so here is a drawing of all the *real points* of M, i.e. the set

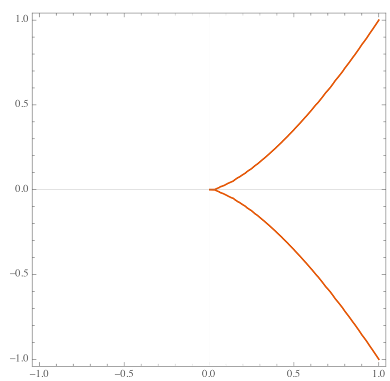
$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : f(x_1, x_2, x_3) = 0\}.$$

This is a subset of $\mathbb{R}^3$, and we can therefore draw it:



Notice that the real points of M are a compact set, but M itself is non-compact. (One can check that it is not bounded, which implies non-compact. This is left as an exercise.) This variety has no singularities, i.e. $\text{sing}(M) = \emptyset$. If (x_1, x_2, x_3) was a singularity, then $f(x_1, x_2, x_3) = 0$ and $(\nabla f)(x_1, x_2, x_3) = 0$. We have $(\nabla f)(x_1, x_2, x_3) = (2x_1, 2x_2, 2x_3)$. So $(\nabla f)(x_1, x_2, x_3) = 0$ implies $(x_1, x_2, x_3) = (0, 0, 0)$. But this point does not satisfy $f(x_1, x_2, x_3) = 0$, so there is no singular point.

Example 3.4. Let $f = x_1^2 - x_2^3$. Then $M := Z(f)$ is a subset of $\mathbb{C}^3$. Again, these are more complex numbers than we can draw, so here is a drawing of the real points of M:



One can check that $\text{sing}(M) = (0, 0)$. This is left as an exercise.

Now comes the definition of *tangent space*:

Definition 3.5. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial and write $M := Z(f)$. For $p \in \text{reg}(M)$ we define

$$N_pM := \text{span}\{(\nabla f)(p)\},$$
$$T_pM := (N_pM)^\perp.$$

The set N_pM is called *normal space of M in p* and the set T_pM is called *tangent space of M in p*.

Lemma 3.6.

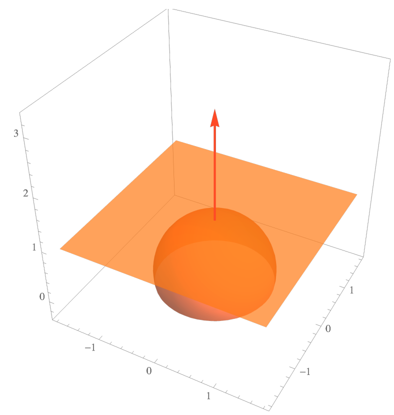
1. *The set N_pM is a complex vector space of dimension 1 and the set T_pM is a complex vector space of dimension $n-1$.*

2. *We have that*
$$T_pM = \{\gamma'(0) \in \mathbb{C}^n : \gamma : (-\epsilon, \epsilon) \to M \text{ is a differentiable curve}\}.$$

Example 3.7. In Example 3.3 we looked at the variety $M := Z(f)$ for $f = x_1^2 + x_2^2 + x_3^2 - 1$. Consider the point $p := (0, 0, 1) \in M$. (This is often called *North Pole*.) We have that $(\nabla f)(p) = (0, 0, 2)$ and therefore $N_p M = \text{span}\{(0, 0, 2)\}$. Consequently, the tangent space is

$$T_p M = \text{span}\{(0, 0, 2)\}^{\perp} = \text{span}\{(1, 0, 0), (0, 1, 0)\}.$$

Here is a picture of this situation: the normal space is the orange arrow pointing upward. The tangent space is the orthogonal complement, it is the orange plane. As expected from Lemma 3.6: the normal space is a 1-dimensional vector space, while the tangent space is a 2-dimensional vector space.



These pictures must be interpreted carefully, because $N_p M$ and $T_p M$ are *complex* vector spaces, but our drawing only shows the *real* vectors. For example, $(0, 0, i)$ is a normal vector, and $(i, 0, 0)$ is a tangent vector, but neither appear appear in the drawing, because the vectors have imaginary entries.

4 Projective Varieties

In the previous section we defined *affine varieties*, but Calabi-Yau manifolds are more complicated objects called *projective varieties*. *Projective* means being a subset of projective space. To this end, we first define projective space and then varieties therein. References are: [Gathmann, 2021, Chapter 5], [Holme, 2011, Section I.1].

We begin with the definition of projective space:

Definition 4.1. On $\mathbb{C}^{n+1} \setminus \{0\}$ define the equivalence relation $\sim$ as follows: $x \sim y$ for $x, y \in \mathbb{C}^{n+1} \setminus \{0\}$ if and only if there exists $\lambda \in \mathbb{R}$ such that $x = \lambda \cdot y$. The set of equivalence classes

$$\mathbb{CP}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\sim$$

is called *n-dimensional complex projective space*. We write

$$[x_0 : \cdots : x_n]$$

for the equivalence class of $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$.

Note that by definition $[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n]$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

The set $\mathbb{CP}^n$ is not just a set, but it has extra structures. It is a topological space, and even a smooth manifolds, but we will not define these terms here because we do not need them.

Important for us are the following subsets of projective space:

Definition 4.2. For $i \in \{0, \ldots, n\}$ the set

$$U_i := \{[x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n] \in \mathbb{CP}^n\} \subset \mathbb{CP}^n$$

is called *the i-th affine chart of* $\mathbb{CP}^n$.

Lemma 4.3.

1. *For $i \in \{0,\dots,n\}$ the map*

$$\phi_i : \mathbb{C}^n \to U_i$$
$$(x_1,\dots,x_n) \mapsto [x_1 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n]$$

is bijective.

2. *For $i \in \{0,\dots,n\}$ we have*

$$\mathbb{CP}^n \setminus U_i = \{[x_0 : \cdots : x_{i-1} : 0 : x_{i+1} : \cdots : x_n] \in \mathbb{CP}^n\}.$$

We leave the proof as an exercise. Next comes the definition of varieties in projective space. While every polynomial defined an affine variety, in projective space the polynomial only has a well-defined zero locus if it is homogeneous.

Definition 4.4. Denote by $\mathbb{C}[x_0,\dots,x_n]$ the set of polynomials in variables $x_0,\dots,x_n$ with complex coefficients. A polynomial f is called *homogeneous of degree* k if all monomials of f have degree k.

Example 4.5. The polynomial $f \in \mathbb{C}[x_0,x_1,x_2]$ given by $f = x_0^3 + x_1^2 x_2 + x_2^3$ has the monomials x_0^3 and $x_1^2 x_2$ and x_2^3. Every monomial has degree 3, so f is homogeneous.
The polynomial $g \in \mathbb{C}[x_0,x_1,x_2]$ given by $g = x_0^2 + x_1$ has the monomials x_0^2 and x_1. The two monomials have degrees 2 and 1 respectively, so g is not homogeneous.

Definition 4.6. Let $f \in \mathbb{C}[x_0,\dots,x_n]$ be a homogeneous polynomial. The set

$$Z(f) := \{[x_0 : \cdots : x_n] \in \mathbb{CP}^n : f(x_0,\dots,x_n) = 0\}$$

is called the *zero locus of f* or *vanishing locus of f*.

It is sometimes denoted by $V(f)$ instead of than $Z(f)$.

Lemma 4.7. *For a homogeneous polynomial $f \in \mathbb{C}[x_0,\dots,x_n]$ the set $Z(f)$ is well-defined.*

Proof. The set is defined as all equivalence classes $[x_0 : \cdots : x_n] \in \mathbb{CP}^n$ satisfying $f(x_0,\dots,x_n) = 0$. We have to check that this definition is independent of the representative of the equivalence class $[x_0 : \cdots : x_n]$. Let $(\lambda x_0,\dots,\lambda x_n) \in [x_0 : \cdots : x_n]$ for $\lambda \neq 0$ be a second representative of the same equivalence class. Let d be the degree of f. Then, because f is homogeneous:

$$f(\lambda x_0,\dots,\lambda x_n) = \lambda^d f(x_0,\dots,x_n).$$

Therefore, $f(\lambda x_0,\dots,\lambda x_n) = 0$ if and only if $f(x_0,\dots,x_n)$, which shows that the condition $f(x_0,\dots,x_n) = 0$ is independent of the chosen representative. □

Remark 4.8. Let $f \in \mathbb{C}[x_0,\dots,x_n]$ be a homogeneous polynomial and $M := Z(f)$. For $i \in \{0,\dots,n\}$ let $\phi_i : \mathbb{C}^n \to U_i$ be the map from Lemma 4.3. Then $\phi_i^{-1}(X \cap U_i)$ is an affine variety given by equation

$$\{(x_1,\dots,x_n) \in \mathbb{C}^n : f(x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_n) = 0\}.$$

Example 4.9. We revisit Example 3.3. This time define $f = -x_0^2 + x_1^2 + x_2^2 + x_3^2$ and $M := Z(f)$, which is then a subset of $\mathbb{CP}^3$. If we take $\phi_i^{-1}(X \cap U_i)$ for $i = 0$, as explained in Remark 4.8, then we obtain the variety $Z(-1^2 + x_1^2 + x_2^2 + x_3^2) \subset \mathbb{C}^3$. We got this just by setting $x_0 = 1$ in the polynomial f.
This is now an affine variety, the polynomial $-1^2 + x_1^2 + x_2^2 + x_3^2$ is *not homogenous*. It is, in fact, exactly the affine variety from Example 3.3.

Remark 4.10. When people draw pictures of projective varieties, here is what they *really* do: first, restrict to one *affine* chart. Second, draw only the *real* points of the variety. For example, the picture in Example 3.3 would represent the projective variety $M := Z(f)$ for $f = -x_0^2 + x_1^2 + x_2^2 + x_3^2$. Note that this such a picture does *not* capture all the information about the projective variety. There will be many projective varieties which have the same affine, real drawing.

5 Riemannian metrics

In this section we will define Riemannian metrics only for affine varieties. They can be defined in much more generality, but for this one would need to define *manifolds*, which we have avoided in this text. References are: [O'neill, 2006, Chapter 7], [Bär, 2013, Section 2.2]. Note that the latter reference makes very general definitions, that requires the abstract definition of manifolds, while to former does not.
Here is the definition:

Definition 5.1. Let $M = Z(f) \subset \mathbb{C}^n$ be a smooth affine variety. A *Hermitian metric h* is a map

$$h : M \to \bigcup_{p \in M} \operatorname{Bil}(T_p M) \tag{5.2}$$

with the property that $h(p) \in \operatorname{Bil}(T_p M)$. A *Riemannian metric g* is a map

$$g : M \to \bigcup_{p \in M} \operatorname{Bil}((T_p M)_{\mathbb{R}}) \tag{5.3}$$

with the property that $g(p) \in \operatorname{Bil}((T_p M)_{\mathbb{R}})$, where $(T_p M)_{\mathbb{R}}$ denotes the vector space $T_p M$ of complex dimension $n-1$ viewed as a real vector space of dimension $2(n-1)$.

Remark 5.4. Normally one also requires the map from Eq. (5.3) to be continuous. However, we have not defined a topology on the target space $\bigcup_{p \in M} \operatorname{Bil}(T_p M)$, so we cannot make this definition here.

Example 5.5 (The induced metrics). An affine variety $M = Z(f) \subset \mathbb{C}^n$ always carries a standard metric induced by the standard Euclidean inner product induced by Example 2.3. Explicitly, this means

$$g(p)(v, w) := r(v)^T r(w),$$

where $r : \mathbb{C}^n \to \mathbb{R}^{2n}$ is the reellification map.

6 Calabi-Yau metrics

In this last section we mention Calabi-Yau manifolds without explaining any of the details. References are: [Yau, 2008], which surveys all the hard maths around Calabi-Yau manifolds and gives many references; [He, 2021], which contains less mathematical detail and discusses connections to machine learning and physics.
Given a Riemannian metric g on M, one can define in each point $p \in M$ an object called *Ricci curvature*, denoted as Ric_p, which is a map

$$\operatorname{Ric}_p : T_p M \times T_p M \to \mathbb{R}.$$

This depends on the metric g! Different metrics almost always give different Ricci tensors.
A Riemannian metric g on a variety is called *Calabi-Yau metric* if $\operatorname{Ric}_p(v, w) = 0$ for all $p \in M$ and $v, w \in T_p M$. These metrics play an important role in string theory as well as in pure mathematics. For a long time their existence could not be proven and explicit formulae for these metrics are very complicated or may not even exist. Surprisingly, for projective varieties there is a necessary and sufficient criterion for admitting a Calabi-Yau metric. This criterion was conjectured by Eugenio Calabi and many years later proved by Shing-Tung Yau. Due to this criterion, there are billions of examples of manifolds known which admit a Calabi-Yau metric, even though no explicit formulae for them exist.

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