# $G_2$-instantons on Resolutions of $G_2$-orbifolds

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## Statement of Originality

The work is my own except where indicated otherwise.

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## Abstract

The resolution of the $G_2$-orbifold $T^7/\Gamma$, where $\Gamma$ is a suitably chosen finite group, admits a 1-parameter family of $G_2$-structures with small torsion $\varphi^t$, obtained by gluing in Eguchi-Hanson spaces. It was shown in [Joy96b] that $\varphi^t$ can be perturbed to a torsion-free $G_2$-structure $\widetilde{\varphi}^t$ for small values of $t$. Using norms adapted to the geometry of the manifold we give an alternative proof of the existence of $\widetilde{\varphi}^t$. This alternative proof produces the estimate $\left\|\widetilde{\varphi}^t - \varphi^t\right\|_{C^0} \leq ct^{5/2}$. This is an improvement over the previously known estimate $\left\|\widetilde{\varphi}^t - \varphi^t\right\|_{C^0} \leq ct^{1/2}$. As part of the proof, we show that Eguchi-Hanson space admits a unique (up to scaling) harmonic form with decay, which is a result of independent interest.

More generally, there exists a construction of torsion-free $G_2$-structures on resolutions of a more general class of $G_2$-orbifolds, given in [JK21]. We explain a construction of $G_2$-instantons on these manifolds, which includes the case of $G_2$-instantons on resolutions of $T^7/\Gamma$ as a special case. The ingredients needed are a $G_2$-instanton on the orbifold and a Fueter section over the singular set of the orbifold. In the general case, we make the very restrictive assumption that the Fueter section is pointwise rigid. In the special case of resolutions of $T^7/\Gamma$, the improved estimate for $\widetilde{\varphi}^t - \varphi^t$ allows to remove this assumption. As an application, we construct one new example of a $G_2$-instanton on the resolution of $(T^3 \times \mathrm{K3})/\mathbb{Z}_2^2$.

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# 1 Introduction

In [Ber55], Berger presented a list of groups which can possibly occur as the holonomy groups of Riemannian manifolds. However, constructing manifolds which realise these holonomy groups remained a wide-open problem for decades. A milestone in this direction was the formulation and proof of the Calabi conjecture in [Cal54, Cal57] and [Yau77, Yau78] respectively. Among other things, the proof of this conjecture gives a powerful characterisation of manifolds admitting a metric with holonomy $\mathrm{SU}(n)$, giving rise to a wealth of examples of such manifolds. For the exceptional holonomy group $G_2$, such a general characterisation remains out of reach, and even the construction of examples persists to be a challenging task.

The first compact examples of Riemannian manifolds with holonomy equal to $G_2$ were constructed in [Joy96b] by resolving an orbifold of the form $T^7/\Gamma$, where $\Gamma$ is a finite group of isometries of $T^7$. In [JK21], this construction was extended to resolutions of orbifolds of the form $Y/\Gamma$, where $Y$ is a manifold with holonomy contained in $G_2$, but not necessarily flat, and $\Gamma$ is a finite group of $G_2$-involutions. In [Joy96b] and [JK21] this was done by constructing $G_2$-structures with small torsion, and subsequently perturbing them to torsion-free $G_2$-structures. This perturbation made use of a general existence result for torsion-free $G_2$-structures that holds on all 7-manifolds. An immediate question is: how far away is the torsion-free $G_2$-structure from the $G_2$-structure with small torsion? This is important in applications, such as the construction of associative submanifolds and $G_2$-instantons. In Section 3 we give a partial answer to this question by proving an improved estimate for the difference between the torsion-free $G_2$-structure and the one with small torsion for the $G_2$-manifolds from [Joy96b]. The main result of this section is Theorem 3.84:

**Theorem**. *Choose $\alpha \in (0,1)$ and $\beta \in (-1,0)$ both close to 0. Let $N_t$ be the resolution of $T^7/\Gamma$ from Eq. (3.31) and $\varphi^t \in \Omega^3(N_t)$ the $G_2$-structure with small torsion from Eq. (3.33). There exists $c > 0$ independent of $t$ such that the following is true: for $t$ small enough, there exists $\eta^t \in \Omega^2(N_t)$ such that $\widetilde{\varphi} = \varphi^t + \mathrm{d}\eta^t$ is a torsion-free $G_2$-structure, and $\eta^t$ satisfies*

$$\|\eta^t\|_{C^{2,\alpha/2}_{\beta;t}} \leq c t^{7/2-\beta}.$$

*In particular,*

$$\left\|\widetilde{\varphi}-\varphi^{t}\right\|_{L^{\infty}} \leq ct^{5/2} \textit{ and } \left\|\widetilde{\varphi}-\varphi^{t}\right\|_{C^{0,\alpha/2}} \leq ct^{5/2-\alpha/2} \textit{ as well as } \left\|\widetilde{\varphi}-\varphi^{t}\right\|_{C^{1,\alpha/2}} \leq ct^{3/2-\alpha/2}.$$

Here, the norm $||\cdot||_{C^{2,\alpha/2}_{\beta;t}}$ is a weighted Hölder norm. The norms in the last line of the theorem are ordinary, unweighted norms. The group $\Gamma$ is a finite group acting through $G_2$-involutions on $T^7$. In [Joy96b, Joy00] the estimate $||\widetilde{\varphi}-\varphi||_{L^\infty} \leq ct^{1/2}$ was shown. In this sense, the estimates from Theorem 3.84 are an improvement. The theorem hinges on an estimate for the inverse of the Laplacian acting on 2-forms on the resolution of $T^7/\Gamma$. The crucial idea necessary for obtaining this estimate is to split 2-forms into a part that is harmonic on the 4-dimensional fibres orthogonal to the singular set of $T^7/\Gamma$, and a rest. The 4-dimensional fibres are subsets of Eguchi-Hanson space $X_{\mathrm{EH}}$, and the proof of Theorem 3.84 uses detailed knowledge of the harmonic forms on $X_{\mathrm{EH}}$. The space $X_{\mathrm{EH}}$ admits a harmonic 2-form $\nu_1$ that can be written down explicitly and comes from rescaling the metric. In Theorem 3.26, we denote the Laplacian on $X_{\mathrm{EH}}$ acting on $p$-forms by $\Delta_{p,g_{(1)}}$, and we prove that $\nu_1$ is essentially the only form with decay:

**Theorem.** *For $\lambda \in (-4,0)$, the $L^2_{2,\lambda}$-kernels of $\Delta_{p,g_{(1)}}$ acting on $p$-forms of different degrees are the same as the $L^2$-kernels, namely:*

$$\begin{aligned}
\mathrm{Ker}(\Delta_{g_{(1)}} : L^2_{2,\lambda}(\Lambda^2(X_{EH})) \to L^2_{0,\lambda-2}(\Lambda^2(X_{EH}))) &= \langle \nu_1 \rangle, \\
\mathrm{Ker}(\Delta_{g_{(1)}} : L^2_{2,\lambda}(\Lambda^p(X_{EH})) \to L^2_{0,\lambda-2}(\Lambda^p(X_{EH}))) &= 0 \textit{ for } p \neq 2.
\end{aligned}$$

Here $L^2_{2,\lambda}(\Lambda^p(X_{\mathrm{EH}}))$ denote the usual weighted Sobolev spaces on asymptotically conical manifolds. They consist of, roughly speaking, $L^2$-sections with 2 weak derivatives that decay like $r^\lambda$ as $r \to \infty$, where $r$ is a radius function.

Using the idea from [Joy96b], some millions of $G_2$-manifolds can be constructed, see [Joy00, p.322]. However, using Betti numbers alone, only around 100 of them can be distinguished. This prompts the question: how many of these $G_2$-structures are deformation equivalent? An idea that may potentially help to answer this question comes from gauge theory: in the seminal article [Don83], the moduli space of anti-self-dual connections was used to define invariants of smooth 4-manifolds. Following this, a rich theory of gauge theoretical invariants and their relations to other manifold invariants in 4 dimensions was developed. The article [DT98] then recognised some of the 4-dimensional phenomena in dimension 7, for example the existence of a functional whose critical points are instantons. With great optimism, one may hope to recreate the four-dimensional success story in dimension 7, and use the moduli space of $G_2$-instantons to define deformation invariants of $G_2$-manifolds. There are analytic difficulties present in dimension 7 that were not there in dimension 4, and therefore the study of $G_2$-instantons has mainly focused on the construction of examples. The examples that have appeared in the literature so far are [Wal13a, SEW15, Wal16, MNSE21, LO20, LO18]. In Section 4 we add to this as follows: we prove a gluing theorem that can be used to construct $G_2$-instantons on the $G_2$-manifolds from [JK21]. Such a manifold is a resolution of a $G_2$-orbifold, obtained by taking the quotient of a $G_2$-manifold $Y$ by a $G_2$-involution $\iota$. The resolution $N$ is obtained by gluing Eguchi-Hanson spaces over the singular set of $Y/\langle\iota\rangle$. Given a $G_2$-instanton $\theta$ on $Y/\langle\iota\rangle$ one may be able to construct from it a $G_2$-instanton on $N$. To do this, one needs a connection over the glued in part. One way to get such a connection is by taking a suitable family of anti-self-dual instantons over Eguchi-Hanson space, say $s$. Our main result is that one can glue together $\theta$ and $s$ to a genuine $G_2$-instanton if $s$ consists of a rigid instanton in each fibre and they satisfy a simple compatibility condition (cf. Theorem 4.130):

**Theorem.** *Assume now that the section $s$ is given by a rigid ASD-instanton in every point $x \in L$, and assume that the connection $\theta$ used to define the approximate $G_2$-instanton $A_t$ from Proposition 4.27 is infinitesimally rigid.*

*There exists $c > 0$ such that for small $t$ there exists $\underline{a}_t = (a_t, \xi_t) \in C^{1,\alpha}(\Omega^0 \oplus \Omega^1(\operatorname{Ad} E_t))$ such that $\widetilde{A_t} := A_t + a_t$ is a $G_2$-instanton. Furthermore, $\underline{a}_t$ satisfies $\left\|\underline{a}_t\right\|_{C^{1,\alpha}_{-1,\delta;t}} \leq ct^{1/18}$.*

Here, $\alpha \in (0,1)$ must be a small number and $||\cdot||_{C^{1,\alpha}_{-1,\delta;t}}$ denotes a weighted Hölder norm. We use this theorem to construct a new $G_2$-instanton on the resolution of $(T^3 \times \mathrm{K3})/\mathbb{Z}_2^2$.

Thanks to the improved estimate for the difference $\widetilde{\varphi} - \varphi^t$ on resolutions of $T^7/\Gamma$ from the aforementioned Theorem 3.84 we have an even stronger gluing theorem on these manifolds. In this case, we need not require that the section $s$ is given by rigid instantons, only that it is a rigid solution of the Fueter equation (cf. Theorem 4.131):

**Theorem.** *Let $N \to Y'$ be the resolution of the orbifold $Y' = T^7/\Gamma$ from before. Assume that the connection $\theta$ used to define the approximate $G_2$-instanton $A_t$ from Proposition 4.27 is infinitesimally rigid and that $s$ is an infinitesimally rigid Fueter section.*

*There exists $c > 0$ such that for small $t$ there exists an $\underline{a}_t = (a_t, \xi_t) \in C^{1,\alpha}(\Omega^0 \oplus \Omega^1(\operatorname{Ad} E_t))$ such that $\widetilde{A_t} := A_t + a_t$ is a $G_2$-instanton. Furthermore, $\underline{a}_t$ satisfies $\left\|\underline{a}_t\right\|_{\mathfrak{X}_t} \leq ct^{2-2\alpha}$.*

Here, $|| \cdot ||_{\mathfrak{X}_t}$ denotes a complicated composite norm. The basic idea of this norm is the same as in the previous chapter: it consists of a part that is harmonic in the Eguchi-Hanson directions in the gluing region and a rest, and the two parts are scaled differently.

Unfortunately, no genuine examples of these more general ingredients are known. That is: all known rigid Fueter sections are actually sections of rigid instantons. Therefore, we were unable to use this theorem to produce new examples so far.

# 2 Background

## 2.1 Riemannian Holonomy Groups

Let $(M, g)$ be a smooth, $n$-dimensional Riemannian manifold and denote its Levi-Civita connection by $\nabla$.

*Definition* 2.1. Given a piecewise smooth curve $\gamma : [0,1] \to M$ from $\gamma(0) = x$ to $\gamma(1) = y$, denote the parallel transport induced by $\nabla$ along $\gamma$ by $\mathcal{P}_\gamma : T_xM \to T_yM$. For $p \in M$ we then define the *holonomy group of $g$ at $p$* as

$$\mathrm{Hol}(g, p) = \{\mathcal{P}_\gamma : \gamma \text{ smooth loop based at } p\} \subset \mathrm{End}(T_pM).$$

The following are standard properties of holonomy groups, see e.g. [KN63, Chapters II and IV]:

**Lemma 2.2.** 1. *The groups* $\mathrm{Hol}(g, p)$ *and* $\mathrm{Hol}(g, q)$ *are isomorphic groups for all* $p, q \in M$.

2. *For all* $p$ *we have that* $\mathrm{Hol}(g, p)$ *preserves the metric on* $T_pM$, *i.e.* $\mathrm{Hol}(g, p) \subset O(T_pM)$.

Because of the this, we can fix a point $p \in M$ and an isometry $T_pM \simeq \mathbb{R}^n$ and speak of $\mathrm{Hol}(g, p)$ as a subgroup of $O(n)$ and call it the *holonomy group of* $(M, g)$, denoted by $\mathrm{Hol}(g)$.

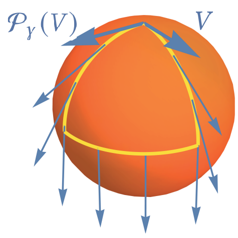


Figure 1: Parallel transport on the sphere $S^2 \subset \mathbb{R}^3$ endowed with the round metric. The tangent vector $V$ is transported along the yellow curve, resulting in the vector $\mathcal{P}_\gamma(V)$. The holonomy group of $S^2$ endowed with the round metric is $\mathrm{SO}(2)$.

Not every Lie group can appear as the holonomy group of a Riemannian manifold. A list of possible holonomy groups was given by Berger:

**Theorem 2.3** ([Ber55]). *Suppose $(M, g)$ is a simply-connected manifold of dimension $n$ that is irreducible and nonsymmetric. Then exactly one of the following holds:*

1. $\mathrm{Hol}(g) = \mathrm{SO}(n)$,
2. $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{U}(m) \subset \mathrm{SO}(2m)$,
3. $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{SU}(m) \subset \mathrm{SO}(2m)$,
4. $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m) \subset SO(4m)$,
5. $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)\,\mathrm{Sp}(1) \subset SO(4m)$,
6. $n = 7$ *and* $\mathrm{Hol}(g) = G_2 \subset \mathrm{SO}(7)$,
7. $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$.

The list originally also included the group $\mathrm{Spin}(9)$, but it was shown in [Ale68] and independently in [BG72] to only occur in symmetric spaces. Berger did not prove that all these groups occur as holonomy groups of Riemannian manifolds, and it took a long time to find example manifolds for each group. In the cases $G_2$ and $\mathrm{Spin}(7)$, metrics with these holonomy groups were shown to exist on non-complete Riemannian manifolds in [Bry87]. The next step was the construction of complete noncompact examples in [BS89]. Finally, compact manifolds with these holonomy groups were constructed in [Joy96b, Joy96a]. In the rest of this section, we will introduce the holonomy groups $\mathrm{Sp}(m)$ and $G_2$ in detail. A thorough discussion of all holonomy groups can be found in [Sal89].

## 2.2 Hyperkähler Geometry and the Eguchi-Hanson Space

We now turn to the holonomy group $\mathrm{Sp}(m)$, the holonomy group of Hyperkähler manifolds. Because of our later applications, we will be particularly interested in dimension four, that is the group $\mathrm{Sp}(1)$.

To this end, consider the blowup of $\mathbb{C}^2/\{\pm 1\}$, which is again a complex surface. More than that, it admits a Hyperkähler structure that is asymptotically locally Euclidean (ALE), see [Joy00, Section 7.2] and [Dan99] for surveys listing these and more properties. In this section, we will define ALE Hyperkähler manifolds, write down an explicit formula for the Hyperkähler metric on the blowup of $\mathbb{C}^2/\{\pm 1\}$ (cf. Proposition 2.5), and show that it satisfies the ALE Hyperkähler property (cf. Proposition 2.10).

We begin with the definition of Hyperkähler manifolds.

*Definition* 2.4. Define the quaternions $\mathbb{H}$ to be the associative, nonabelian real algebra

$$\mathbb{H} = \{x_0 + x_1 i + x_2 j + x_3 k : x_j \in \mathbb{R}\} \simeq \mathbb{R}^4,$$

endowed with the unique multiplication satisfying

$$ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j, \qquad i^2 = j^2 = k^2 = -1.$$

Let $\mathbb{H}^m$ have coordinates $(q^1, \ldots, q^m)$, with $q^l = x^l_0 + x^l_1 i + x^l_2 j + x^l_3 k \in \mathbb{H}$ and $x^l_s \in \mathbb{R}$. Define a metric and 2-forms on $\mathbb{H}^m$ by

$$g = \sum_{l=1}^{m} \sum_{s=0}^{3} (\mathrm{d}x^l_s)^2, \qquad \omega_1 = \sum_{l=1}^{m} \mathrm{d}x^l_0 \wedge \mathrm{d}x^l_1 + \mathrm{d}x^l_2 \wedge \mathrm{d}x^l_3,$$

$$\omega_2 = \sum_{l=1}^{m} \mathrm{d}x^l_0 \wedge \mathrm{d}x^l_2 + \mathrm{d}x^l_3 \wedge \mathrm{d}x^l_1, \qquad \omega_3 = \sum_{l=1}^{m} \mathrm{d}x^l_0 \wedge \mathrm{d}x^l_3 + \mathrm{d}x^l_1 \wedge \mathrm{d}x^l_2.$$

Define complex structures $I, J, K$ on $\mathbb{H}^m$ to be left multiplication with $i, j, k$ respectively. The subgroup of $\mathrm{GL}(4m, \mathbb{R})$ preserving $g, \omega_1, \omega_2, \omega_3$ is $\mathrm{Sp}(m)$. It also preserves $I, J, K$.

A $4m$-dimensional Riemannian manifold $(M, g)$ is called *Hyperkähler* if $\mathrm{Hol}(g) \subset \mathrm{Sp}(m)$.

Thus, on a Hyperkähler manifold we have the data of a metric and three compatible complex structures and symplectic forms. Conversely, a metric together with three parallel symplectic structures that are compatible in this sense defines a Hyperkähler structure on a manifold.

We will now define the Eguchi-Hanson space and the Eguchi-Hanson metrics, which are a 1-dimensional family of Hyperkähler metrics, controlled by a parameter $k \in \mathbb{R}_{\geq 0}$. For $k > 0$ we get a metric on a smooth 4-manifold (this is point one of the following proposition), and for $k = 0$ we get the standard metric on $\mathbb{H}/\{\pm 1\}$ or equivalently $\mathbb{C}^2/\{\pm 1\}$ (this is point two of the following proposition).

**Proposition 2.5.** *Let $r$ be a coordinate on the $\mathbb{R}_{\geq 0}$-factor of $\mathbb{R}_{\geq 0} \times \mathrm{SO}(3)$. Let*

$$\eta^1 = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \eta^2 = 2\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \eta^3 = 2\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3)$$

*and denote the dual basis extended to left-invariant 1-forms on $\mathrm{SO}(3)$ by the same symbols. For $k \geq 0$, let $f_k : \mathbb{R}_{>0} \times \mathrm{SO}(3) \to \mathbb{R}_{>0}$ be defined by $f_k(r) = (k + r^2)^{1/4}$ and set*

$$\mathrm{d}t = f_k^{-1}(r)\,\mathrm{d}r, \qquad e^1(r) = r f_k^{-1}(r)\eta^1, \qquad e^2(r) = f_k(r)\eta^2, \qquad e^3(r) = f_k(r)\eta^3.$$

*Define $\omega_1^{(k)}, \omega_2^{(k)}, \omega_3^{(k)} \in \Omega^2(\mathbb{R}_{>0} \times \mathrm{SO}(3))$ to be*

$$\omega_1^{(k)} = \mathrm{d}t \wedge e^1 + e^2 \wedge e^3, \qquad \omega_2^{(k)} = \mathrm{d}t \wedge e^2 + e^3 \wedge e^1, \qquad \omega_3^{(k)} = \mathrm{d}t \wedge e^3 + e^1 \wedge e^2, \tag{2.6}$$

*and denote by $g_{(k)}$ the metric on $\mathbb{R}_{>0} \times \mathrm{SO}(3)$ that makes $(\mathrm{d}t, e^1, e^2, e^3)$ an orthonormal basis.*

1. *If $k > 0$, consider the copy of $\mathrm{SO}(2)$ in $\mathrm{SO}(3)$ defined by $\{\exp(s \cdot \eta^1) : s \in \mathbb{R}\}$, defining a right action of $\mathrm{SO}(2)$ on $\mathrm{SO}(3)$. Denote by $V \simeq \mathbb{R}^2$ the standard representation of $\mathrm{SO}(2)$. Define $\Psi : \mathrm{SO}(3) \times \mathbb{R}_{>0} \to \mathrm{SO}(3) \times V$ as $\Psi(g, r) = (g, (r, 0))$. Denote*

   $$X_{EH} = \mathrm{SO}(3) \times_{\mathrm{SO}(2)} V.$$

   *Then $\Psi$ induces a smooth injective map $\hat{\Psi} : \mathrm{SO}(3) \times \mathbb{R}_{>0} \to X_{EH}$ that is a diffeomorphism onto its image, and the forms $\hat{\Psi}_*(\omega_i^{(k)})$ can be extended to smooth 2-forms on all of $X_{EH}$. Furthermore, $\hat{\Psi}_*(g_{(k)})$ can also be extended to a metric on all of $X_{EH}$, and $(X_{EH}, \hat{\Psi}_*(g_{(k)}))$ is a Hyperkähler manifold.*

2. *If $k = 0$: parametrise the quaternions as $x_0 + x_1 i + x_2 j + x_3 k$ with $x_0, x_1, x_2, x_3 \in \mathbb{R}$, embed $S^3 \subset \mathbb{H}$ as the unit sphere, and fix the identification $\phi : S^3/\{\pm 1\} \to \mathrm{SO}(3)$ that maps $x$ onto the map $y \mapsto x \cdot y \cdot x^{-1}$, where we use $S^3/\{\pm 1\} \subset \mathbb{H}/\{\pm 1\}$ and $\cdot$ denotes quaternionic*

*multiplication, for* $x \in S^3/\{\pm 1\} \subset \mathbb{H}/\{\pm 1\}$. *Denote*

$$\begin{aligned} \Phi : \mathrm{SO}(3) \times \mathbb{R}_{>0} &\to \mathbb{H}/\{\pm 1\} \\ (x,t) &\mapsto t \cdot \phi^{-1}(x). \end{aligned}$$

*Then* $\Phi^*\omega_i = \omega_i^{(0)}$ *for* $i \in \{1,2,3\}$ *and* $\Phi^* g = g_{((0)}$, *where* $g, \omega_1, \omega_2, \omega_3 \in \Omega^2(\mathbb{H})$ *are defined as in Definition 2.4.*

By slight abuse of notation, we will denote the extensions of $\omega_i^{(k)}$ for $i \in \{1,2,3\}$ and $g_{(k)}$ to $X_{\mathrm{EH}}$ in the case $k > 0$ by the same symbol, suppressing the pushforward under $\hat{\Psi}$.

*Proof.* For $k > 0$: the fact that $\omega_1^{(k)}, \omega_2^{(k)}, \omega_3^{(k)}, g_{(k)}$ can be extended to all of $X_{\mathrm{EH}}$ was proven, for example, in [LM17, Section 2.4]. One checks using a direct computation that $\omega_i^{(k)}$ for $i \in \{1,2,3\}$ is closed and [Hit87, Lemma 6.8] implies that $\omega_i^{(k)}$ is also parallel for $i \in \{1,2,3\}$. Both the symplectic forms and the metric are defined using the same orthonormal basis, which proves that they are compatible. The case $k = 0$ is a direct calculation. □

*Remark* 2.7. A possible point of confusion is that the function $r : X_{\mathrm{EH}} \to \mathbb{R}$ is approximately the squared distance to the bolt $\mathrm{SO}(3) \times_{SO(2)} \{0\}$ of $X_{\mathrm{EH}}$, so it is not a radius function.

It is a folklore result that the group of isometries of $X_{\mathrm{EH}}$ that also preserve $J_1^{(k)}$ is isomorphic to $\mathrm{U}(2)/\{\pm 1\}$. This can be seen rather explicitly using the description of the metric from Proposition 2.5, and we give a proof of that in Proposition A.1.

The Hyperkähler structure on $X_{\mathrm{EH}}$ also has the important property that it approximates the flat Hyperkähler structure on $\mathbb{R}^4$ for large values of $r$. The following definition makes this notion precise, and Proposition 2.10 proves that the Hyperkähler structure on $X_{\mathrm{EH}}$ does indeed have this property.

*Definition* 2.8 (Definition 7.2.1 in [Joy00]). Let $G$ be a finite subgroup of $\mathrm{Sp}(1)$, and let $(\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3, \hat{g})$ be the Euclidean Hyperkähler structure on $\mathbb{H}$, and $\sigma : \mathbb{H}/G \to [0,\infty)$ the radius function on $\mathbb{H}/G$. We say that a Hyperkähler 4-manifold $(X, \omega_1, \omega_2, \omega_3, g)$ is *asymptotically locally Euclidean (ALE) asymptotic to* $\mathbb{H}/G$, if there exists a compact subset $S \subset X$ and a map $\pi : X \setminus S \to$

$\mathbb{H}/G$ that is a diffeomorphism between $X \setminus S$ and $\{x \in \mathbb{H}/G : \sigma(x) > R\}$ for some $R > 0$, such that

$$\hat{\nabla}^k(\pi_*(g) - \hat{g}) = O(\sigma^{-4-k}) \text{ and } \hat{\nabla}^k(\pi_*(\omega_i) - \hat{\omega}_i) = O(\sigma^{-4-k}) \tag{2.9}$$

as $\sigma \to \infty$, for $i \in \{1,2,3\}$ and $k \geq 0$, where $\hat{\nabla}$ is the Levi-Civita connection of $\hat{g}$.

**Proposition 2.10.**

1. *The* 2*-sphere* $Y := \mathrm{SO}(3) \times_{\mathrm{SO}(2)} \{0\} \subset X_{EH}$ *has radius* $k^{1/4}$.
2. *There exists* $\tau_1^{(k)} \in \Omega^1(X_{EH} \setminus \mathrm{SO}(3) \times_{\mathrm{SO}(2)} \{0\})$ *such that* $\omega_1^{(k)} - \omega_1^{(0)} = \mathrm{d}\tau_1^{(k)}$ *and for any* $l \in \mathbb{Z}$

$$\left|\nabla^l \tau_1^{(1)}\right|_{g_{(0)}} = O(r^{-3-l}), \tag{2.11}$$

*where* $\nabla$ *denotes the Levi-Civita connection of* $g_{(0)}$. *Furthermore,* $\omega_2^{(k)} - \omega_2^{(0)} = 0$, *and* $\omega_3^{(k)} - \omega_3^{(0)} = 0$. *In particular,* $(X_{EH}, \omega_1^{(k)}, \omega_2^{(k)}, \omega_3^{(k)}, g_{(k)})$ *is ALE asymptotic to* $\mathbb{H}/\{\pm 1\}$.

3. *For* $k, k' > 0$ *there exists a diffeomorphism* $\phi_{k,k'} : X_{EH} \to X_{EH}$ *s.t.* $\phi^*_{k,k'}(g_{(k)}) = \lambda^2 g_{(k')}$ *for* $\lambda^4 = \frac{k}{k'}$, *which restricts to the identity on* $Y$.

*Proof.*

1. The curve $\gamma(s) = [\exp_{\mathrm{Id}}(s\eta^2), 0]$ is a geodesic in $Y \subset X_{\mathrm{EH}}$ with $\gamma(0) = \gamma(2\pi)$ of length $2\pi k^{1/4}$, so $S^2$ has radius $k^{1/4}$.
2. Explicitly, $\tau_1^{(k)} = (f_k^2 - f_0^2)\eta^1$. The ALE property is [Joyoo, Example 7.2.2].
3. The fact that $g_{(k)}$ and $g_{(k')}$ are conformally equivalent is clear on abstract grounds, as there exists a classification of asymptotically locally Euclidean Hyperkähler metrics (this argument is used in [Joyoo, p. 154]). Explicitly,

$$\begin{aligned} \phi : \mathrm{SO}(3) \times_{\mathrm{SO}(2)} V &\to \mathrm{SO}(3) \times_{\mathrm{SO}(2)} V \\ [u, (r, 0)] &\to \left[u, (\lambda^2 r, 0)\right] \end{aligned} \tag{2.12}$$

satisfies the claim in the proposition.

□

*Remark* 2.13. By definition, $X_{\mathrm{EH}}$ is an associated bundle over $\mathrm{SO}(3)/\mathrm{SO}(2) = S^2$. In fact, $X_{\mathrm{EH}}$ is diffeomorphic to the total space of $T^*S^2$, which itself is diffeomorphic to $T^*\mathbb{CP}^1$. It is a folklore result that $(X_{\mathrm{EH}}, J_1^{(k)})$ is biholomorphic to $T^*\mathbb{CP}^1$ for all $k > 0$, which in turn is the blowup of $\mathbb{C}^2/\{\pm 1\}$ in the origin, see e.g. [Dan99, p. 17] for the statement. We thus have a blowup map $\rho : X_{\mathrm{EH}} \to \mathbb{C}^2/\{\pm 1\}$.

There is another description of the ALE metric on Eguchi-Hanson space arising from two different Hyperkähler quotient constructions: first, $X_{\mathrm{EH}}$ is a special case of the Calabi-Yau metrics on $T^*\mathbb{CP}^n$ explained in [GRG97]. Second, $X_{\mathrm{EH}}$ is a special case of ALE manifolds asymptotic to the metric on $\mathbb{C}^2/\Gamma$, where $\Gamma \subset \mathrm{SU}(2)$ is a finite subgroup, which is explained in [Kro89a]. (The special case of Eguchi-Hanson space in this construction is described in [GN92, Section 2].)

We briefly describe the construction from [GRG97], as it will be needed in Section 2.4.2. Let $\mathcal{M} = \mathbb{H}^2$ with quaternionic coordinates $q_a$, $a \in \{1, 2\}$, and let $\mathrm{U}(1)$ act on $\mathcal{M}$ via

$$q_a \mapsto q_a e^{it}, \quad t \in (0, 2\pi]. \tag{2.14}$$

A Hyperkähler moment map for this action is given by

$$\begin{aligned} \mu : \mathcal{M} &\to \mathrm{Im}(\mathbb{H}) \simeq \mathbb{R}^3 \otimes \mathfrak{u}(1) \\ (q_1, q_2) &\mapsto \frac{1}{2} \sum_{a \in \{1,2\}} q_a i \overline{q}_a. \end{aligned} \tag{2.15}$$

Let $\zeta = \frac{i}{2} \in \mathrm{Im}(\mathbb{H})$. The group $\mathrm{U}(1)$ acts freely on $\mu^{-1}(\zeta)$ and the general theory of Hyperkähler reduction gives rise to a Hyperkähler structure on the four-dimensional manifold $X' = \mu^{-1}(\zeta)/\mathrm{U}(1)$, denoted by $\mathcal{M}/\!\!/\!\!/\mathrm{U}(1)$.

It will turn out that $X'$ and $X_{\mathrm{EH}}$ are isomorphic as Hyperkähler manifolds. We now identify the group of holomorphic isometries of $X'$, thereby recovering the result of Proposition A.1. We view $\mathrm{SU}(2)$ embedded in $\mathbb{H}^{2\times 2}$ as quaternion valued matrices with no $j$ or $k$ components.

Then $\mathrm{SU}(2)$ acts on $\mathcal{M}$ by right multiplication. This action restricts to $\mu^{-1}(\zeta)$ and commutes with the action of $\mathrm{U}(1)$. The action is not effective, as $-1 \in \mathrm{SU}(2)$ acts trivially, but the induced action of the quotient group $\mathrm{SU}(2)/\{\pm 1\} \simeq \mathrm{SO}(3)$ is effective. Next, let $\mathrm{SO}(2)$ act on $\mathcal{M}$ from the left via

$$q_a \mapsto e^{it} \cdot q_a, \quad t \in (0, 2\pi].$$

Again, the action restricts to $\mu^{-1}(\zeta)$ and commutes with the action of $\mathrm{U}(1)$, but is not effective as $-1 \in \mathrm{SO}(2)$ acts trivially. The actions of $\mathrm{SO}(2)/\{\pm 1\}$ and $\mathrm{SU}(2)/\{\pm 1\}$ commute, as the first group is acting from the left, the second is acting from the right. We thus get that the group $\mathrm{SO}(2)/\{\pm 1\} \times \mathrm{SU}(2)/\{\pm 1\}$ acts through isometries on $X'$. Last, one readily confirms that the map

$$\begin{aligned} U(1)/\{\pm 1\} \times \mathrm{SU}(2)/\{\pm 1\} &\to \mathrm{U}(2)/\{\pm 1\} \\ [\lambda], [A] &\mapsto [\lambda A] \end{aligned}$$

is a group isomorphism. Its inverse is given by $[B] \mapsto ([\sqrt{\det B}], [B/\sqrt{\det B}])$ which is not well-defined as a map $\mathrm{U}(1) \times \mathrm{SU}(2) \to \mathrm{U}(2)$ but is well-defined after dividing out $\{\pm 1\}$. One may also recover the full isometry group of the Eguchi-Hanson space by noticing that there is an additional isometry induced by the map on $\mathcal{M}$ that swaps coordinates, i.e. $\mathcal{M} \to \mathcal{M}$, $(q_1, q_2) \mapsto (q_2, q_1)$.

As a smooth manifold, $X' \simeq T^*\mathbb{CP}^1$, so $X_{\mathrm{EH}}$ and $X'$ are diffeomorphic by Remark 2.13. The Hyperkähler metric on $X'$ is asymptotically locally Euclidean by [CGLP01, Section 2.4]. By [Joy96b, Example 7.2.2], $X'$ is isomorphic as a Hyperkähler manifold to $(X_{\mathrm{EH}}, g_{(k)})$ for some $k > 0$. The curve $\gamma : [0, 2\pi] \to X'$ given by

$$(1, 0) \cdot \exp\left( t \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)$$

parametrises a perimeter of the minimal 2-sphere $(1, 0) \cdot \mathrm{SO}(3)$ in $X'$. It has length $2\pi$, so $X'$ is isomorphic to the Hyperkähler manifold $(X_{\mathrm{EH}}, g^{(1)})$ by the first point of Proposition 2.10.

We sum up the results:

**Proposition 2.16.** *Under the* $\mathrm{U}(1)$*-action on* $\mathcal{M} := \mathbb{H}^2$ *from Eq.* (2.14) *we have that* $\mathcal{M}/\!\!/\!\!/\mathrm{U}(1) \simeq (X_{EH}, g^{(1)})$ *as Hyperkähler manifolds.*

## 2.3 $G_2$-structures

### 2.3.1 Torsion of $G_2$-structures on 7-manifolds

We now introduce $G_2$-structures and their torsion, following the treatment in [Joy00].

*Definition* 2.17 (Definition 10.1.1 in [Joy00]). Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$. Write $\mathrm{d}x_{ij\ldots l}$ for the exterior form $\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \cdots \wedge \mathrm{d}x_l$. Define $\varphi_0 \in \Omega^3(\mathbb{R}^7)$ by

$$\varphi_0 = \mathrm{d}x_{123} + \mathrm{d}x_{145} + \mathrm{d}x_{167} + \mathrm{d}x_{246} - \mathrm{d}x_{257} - \mathrm{d}x_{347} - \mathrm{d}x_{356}. \tag{2.18}$$

The subgroup of $\mathrm{GL}(7, \mathbb{R})$ preserving $\varphi_0$ is the exceptional Lie group $G_2$. It also fixes the Euclidean metric $g_0 = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_7^2$, the orientation on $\mathbb{R}^7$, and $*\varphi_0 \in \Omega^4(\mathbb{R}^7)$.

*Definition* 2.19. The skew-symmetric bilinear map $\times : \mathbb{R}^7 \to \mathbb{R}^7$ defined by

$$\varphi_0(u, v, w) = g_0(u \times v, w)$$

for $u, v, w \in \mathbb{R}^7$ is called the *cross product induced by* $\varphi$.

**Theorem 2.20** (Theorem 8.5 in [SW17]). *Let* $\psi = *\varphi_0$. *Then* $\Lambda^*(\mathbb{R}^7)^*$ *splits into irreducible representations of* $G_2$ *as follows:*

$$\begin{aligned}
\Lambda^1 V^* &= \Lambda^1_7, \\
\Lambda^2 V^* &= \Lambda^2_7 \oplus \Lambda^2_{14}, \\
\Lambda^3 V^* &= \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}
\end{aligned}$$

*and correspondingly for* $\Lambda^k(\mathbb{R}^7)^* \simeq \Lambda^{7-k}(\mathbb{R}^7)^*$ *with* $k = 4, 5, 6$. *Here,* $\dim \Lambda^k_d = d$ *and*

$$\Lambda^2_7 := \{\alpha : *(\alpha \wedge \varphi_0) = 2\alpha\} = \{i(u)\varphi_0 : u \in \mathbb{R}^7\} \simeq \Lambda^1_7,$$

$$\Lambda^2_{14} := \{\alpha : *(\alpha \wedge \varphi_0) = -\alpha\} = \{\alpha : \alpha \wedge \psi = 0\} \simeq \mathfrak{g}_2,$$

$$\Lambda^3_1 := \langle \varphi_0 \rangle,$$

$$\Lambda^3_7 := \{i(u)\psi : u \in \mathbb{R}^7\} \simeq \Lambda^1_7, \text{ and}$$

$$\Lambda^3_{27} := \{\alpha : \alpha \wedge \varphi_0 = 0 \text{ and } \alpha \wedge \psi = 0\} \simeq \mathrm{Sym}_0(\mathbb{R}^7)$$

*Definition* 2.21. Let $M$ be an oriented 7-manifold. A principal subbundle $Q$ of the bundle of oriented frames with structure group $G_2$ is called a $G_2$*-structure*. Viewing $Q$ as a set of linear maps from tangent spaces of $M$ to $\mathbb{R}^7$, there exists a unique $\varphi \in \Omega^3(M)$ such that $Q$ identifies $\varphi$ with $\varphi_0 \in \Omega^3(\mathbb{R}^7)$ at every point.

Such $G_2$-structures are in 1-1 correspondence with 3-forms on $M$ for which there exists an oriented isomorphism mapping them to $\varphi_0$ at every point. We will therefore also refer to such 3-forms as $G_2$-structures.

Let $M$ be a manifold with $G_2$-structure $\varphi$. We call $\nabla\varphi$ the *torsion* of a $G_2$-structure $\varphi \in \Omega^3(M)$. Here, $\nabla$ denotes the Levi-Civita induced by $\varphi$ in the following sense: we have $G_2 \subset \mathrm{SO}(7)$, so $\varphi$ defines a Riemannian metric $g$ on $M$, which in turn defines a Levi-Civita connection. As a shorthand, we also use the following notation: write $\Theta(\varphi) = *\varphi$, where "$*$" denotes the Hodge star defined by $g$. Using this, the following theorem gives a characterisation of torsion-free $G_2$-manifolds:

**Theorem 2.22** (Propositions 10.1.3 and 10.1.5 in [Joy00]). *Let $M$ be an oriented 7-manifold with $G_2$-structure $\varphi$ with induced metric $g$. The following are equivalent:*

*(i)* $\mathrm{Hol}(g) \subseteq G_2$,

*(ii)* $\nabla\varphi = 0$ *on $M$, where $\nabla$ is the Levi-Civita connection of $g$, and*

*(iii)* $\mathrm{d}\varphi = 0$ *and* $\mathrm{d}\Theta(\varphi) = 0$ *on $M$.*

*If these hold then $g$ is Ricci-flat.*

The goal of Section 3 will be to construct $G_2$-structures that induce metrics with holonomy *equal* to $G_2$. A torsion-free $G_2$-structure alone only guarantees holonomy *contained* in $G_2$, but in the compact setting a characterisation of manifolds with holonomy equal to $G_2$ is available:

**Theorem 2.23** (Proposition 10.2.2 and Theorem 10.4.4 in [Joy00]). *Let $M$ be a compact oriented manifold with torsion-free $G_2$-structure $\varphi$ and induced metric $g$. Then $\mathrm{Hol}(g) = G_2$ if and only if $\pi_1(M)$ is finite. In this case the moduli space of metrics with holonomy $G_2$ on $M$, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $b^3(M)$.*

Note that this theorem makes no statement about the existence of a torsion-free $G_2$-structure in the first place. Finding a characterisation of manifolds which admit a torsion-free $G_2$-structure and even the construction of examples remain challenging problems in the field.

Later on, we will investigate perturbations of $G_2$-structures and analyse how that changes their torsion. To this end, we will use the following estimates for the map $\Theta$ defined before:

**Proposition 2.24** (Proposition 10.3.5 in [Joy00] and eqn. (21) of part I in [Joy96b]). *There exists $\epsilon > 0$ and $c > 0$ such that whenever $M$ is a 7-manifold with $G_2$-structure $\varphi$ satisfying $\mathrm{d}\varphi = 0$, then the following is true. Suppose $\chi \in C^\infty(\Lambda^3 T^*M)$ and $|\chi| \leq \epsilon$. Then $\varphi + \chi$ is a $G_2$-structure, and*

$$\Theta(\varphi + \chi) = *\varphi - T(\chi) - F(\chi), \tag{2.25}$$

*where "$*$" denotes the Hodge star with respect to the metric induced by $\varphi$, $T : \Omega^3(M) \to \Omega^4(M)$ is a linear map (depending on $\varphi$), and $F$ is a smooth function from the closed ball of radius $\epsilon$ in $\Lambda^3 T^*M$ to $\Lambda^4 T^*M$ with $F(0) = 0$. Furthermore,*

$$\begin{aligned}
|F(\chi)| &\leq c\,|\chi|^2, \\
|\mathrm{d}(F(\chi))| &\leq c\left\{|\chi|^2\,|\mathrm{d}^*\varphi| + |\nabla\chi|\,|\chi|\right\}, \\
[\mathrm{d}(F(\chi))]_\alpha &\leq c\left\{[\chi]_\alpha\,\|\chi\|_{L^\infty}\,\|\mathrm{d}^*\varphi\|_{L^\infty} + \|\chi\|_{L^\infty}^2\,[\mathrm{d}^*\varphi]_\alpha + [\nabla\chi]_\alpha\,\|\chi\|_{L^\infty} + \|\nabla\chi\|_{L^\infty}\,[\chi]_\alpha\right\},
\end{aligned}$$

*as well as*

$$|\nabla(F(\chi))| \leq c\left\{|\chi|^2 |\nabla\varphi| + |\nabla\chi|\,|\chi|\right\},$$
$$[\nabla(F(\chi))]_{C^{0,\alpha}} \leq c\left\{[\chi]_\alpha \|\chi\|_{L^\infty} \|\nabla\varphi\|_{L^\infty} + \|\chi\|^2_{L^\infty} [\nabla\varphi]_\alpha + [\nabla\chi]_\alpha \|\chi\|_{L^\infty} + \|\nabla\chi\|_{L^\infty} [\chi]_\alpha\right\}.$$

*Here,* $|\cdot|$ *denotes the norm induced by* $\varphi$, $\nabla$ *denotes the Levi-Civita connection of the metric induced by* $\varphi$, *and* $[\cdot]_{C^{0,\alpha}}$ *denotes the unweighted Hölder semi-norm induced by this metric.*

Finally, the landmark result on the existence of torsion-free $G_2$-structures is the following theorem. It first appeared in [Joy96b, part I, Theorem A], and we present a rewritten version in analogy with [JK21, Theorem 2.7]:

**Theorem 2.26.** *Let* $\alpha, K_1, K_2, K_3$ *be any positive constants. Then there exist* $\epsilon \in (0,1]$ *and* $K_4 > 0$, *such that whenever* $0 < t \leq \epsilon$, *the following holds.*

*Let* $M$ *be a compact oriented* 7*-manifold, with* $G_2$*-structure* $\varphi$ *with induced metric* $g$ *satisfying* $\mathrm{d}\varphi = 0$. *Suppose there is a closed* 3*-form* $\psi$ *on* $M$ *such that* $\mathrm{d}^*\varphi = \mathrm{d}^*\psi$ *and*

*(i)* $\|\psi\|_{C^0} \leq K_1 t^\alpha$, $\|\psi\|_{L^2} \leq K_1 t^{7/2+\alpha}$, *and* $\|\psi\|_{L^{14}} \leq K_1 t^{-1/2+\alpha}$.

*(ii) The injectivity radius* inj *of* $g$ *satisfies* $\mathrm{inj} \geq K_2 t$.

*(iii) The Riemann curvature tensor* Rm *of* $g$ *satisfies* $\|\mathrm{Rm}\|_{C^0} \leq K_3 t^{-2}$.

*Then there exists a smooth, torsion-free* $G_2$*-structure* $\widetilde{\varphi}$ *on* $M$ *such that* $\|\widetilde{\varphi} - \varphi\|_{C^0} \leq K_4 t^\alpha$ *and* $[\widetilde{\varphi}] = [\varphi]$ *in* $H^3(M, \mathbb{R})$. *Here all norms are computed using the original metric* $g$.

The main purpose of Section 3 will be to prove an improved existence theorem, specialised to the resolution of $T^7/\Gamma$. This will be achieved in Theorem 3.82.

### 2.3.2 $G_2$-manifolds and Hyperkähler 4-manifolds

On $\mathbb{H}$ with coordinates $(y_0, y_1, y_2, y_3)$ we have the three symplectic forms $\omega_1, \omega_2, \omega_3$ from Definition 2.4 given as

$$\omega_0 = \mathrm{d}y_0 \wedge \mathrm{d}y_1 + \mathrm{d}y_2 \wedge \mathrm{d}y_3, \quad \omega_1 = \mathrm{d}y_0 \wedge \mathrm{d}y_2 - \mathrm{d}y_1 \wedge \mathrm{d}y_3, \quad \omega_2 = \mathrm{d}y_0 \wedge \mathrm{d}y_3 + \mathrm{d}y_1 \wedge \mathrm{d}y_2.$$

Identify $\mathbb{R}^7$ with coordinates $(x_1, \ldots, x_7)$ with $\mathbb{R}^3 \oplus \mathbb{H}$ with coordinates $((x_1, x_2, x_3), (y_1, y_2, y_3, y_4))$. Then we have for $\varphi_0, *\varphi_0$ from Definition 2.17:

$$\varphi_0 = \mathrm{d}x_{123} - \sum_{i=1}^{3} \mathrm{d}x_i \wedge \omega_i, \qquad *\varphi_0 = \mathrm{vol}_{\mathbb{H}} - \sum_{\substack{(i,j,k)=(1,2,3) \\ \text{and cyclic permutation}}} \omega_i \wedge \mathrm{d}x_{jk}. \tag{2.27}$$

This linear algebra statement easily extends to product manifolds in the following sense: if $X$ is a Hyperkähler 4-manifold, and $\mathbb{R}^3$ is endowed with the Euclidean metric, then $\mathbb{R}^3 \times X$ has a $G_2$-structure. The $G_2$-structure is given by the same formula as in the flat case, namely Eq. (2.27), after replacing $(\omega_1, \omega_2, \omega_3)$ with the triple of parallel symplectic forms defining the Hyperkähler structure on $X$. This *product $G_2$-structure* will be glued into $G_2$-orbifolds in the following sections.

## 2.4 Gauge Theory in Dimension 4

In this part we briefly review the theory of ASD instantons on compact 4-manifolds as well as the (non-compact) ALE spaces. We follow the treatment of [DK90] for the compact case, and the treatment of [Nak90] for ALE spaces.

Let $(X^4, g)$ be an oriented Riemannian 4-manifold. Let $\Omega^2(X) = \Omega^+(X) \oplus \Omega^-(X)$ be the decomposition of $\Omega^2(X)$ into positive and negative eigenspaces of the Hodge $*$-operator. A connection $A$ on a principal $G$-bundle $P$ is then called an *anti-self-dual instanton* (or *ASD instanton*) if its curvature $F_A$ satisfies $*F_A = -F_A$, where $F_A$ is viewed as an element in $\Omega^2(X, \mathrm{Ad}\, E)$, and $*$ acts on the 2-form part while leaving the $\mathrm{Ad}\, P$ part unchanged.

### 2.4.1 On Compact Manifolds

Now, let $Y$ be a compact 4-manifold.

*Definition* 2.28. Fix some smooth connection $A_0$ on $P$ and assume there exists a faithful representation $V$ of $G$. Write $E = P \times_G V$ and for fixed $l \in \mathbb{N}$, $l \geq 3$, we then define:

$$\begin{aligned}
\mathscr{A}^l_{\text{asd}} &:= \{A_0 + a : a \in L^2_l(\Lambda^1(\operatorname{Ad} P), A_0 + a \text{ is anti-self-dual})\},\\
\mathscr{G}^{l+1} &:= \{s \in L^2_{l+1}(\Lambda^0(\operatorname{End}(E))) : s(y) \in G \text{ for all } y \in Y\},\\
M(l) &:= \mathscr{A}^l_{\text{asd}}/\mathscr{G}^{l+1}.
\end{aligned}$$

Here, $\mathscr{G}^{l+1}$ can be identified with gauge transformations of the bundle $P$, and through this acts on $\mathscr{A}^l_{\text{asd}}$ via pullback. Then, $M := M(3)$ is called the *moduli space of ASD instantons.*

*Remark* 2.29. By the Sobolev Embedding Theorem, equivalence classes in $M(3)$ have continuous representatives. Elements in $\mathscr{A}^l_{\text{asd}}$ need not have continuous representatives for $l \leq 2$, which is the reason for the choice $l \geq 3$ here. On the other hand, Proposition 2.30 states, roughly speaking, that the exact value of $l$ does not matter, as long as it is at least 3.

It is now that we make use of the compactness assumption. If $Y$ is compact, then the definition of $M$ actually turns out to be independent of the chosen regularity $l$ in the following sense:

**Proposition 2.30** (Proposition 4.2.16 in [DK90]). *The natural inclusion of $M(l+1)$ in $M(l)$ is a homeomorphism for $l \geq 3$.*

Because of this proposition, we may think of the moduli space to be made up of *smooth* ASD instantons and *smooth* gauge transformations. Define the operator

$$\begin{aligned}
\delta_A : \Omega^1(Y, \operatorname{Ad} P) &\longrightarrow \Omega^0(Y, \operatorname{Ad} P) \oplus \Omega^2_+(Y, \operatorname{Ad} P)\\
a &\longmapsto (\mathrm{d}^*_A a, \mathrm{d}^+_A a),
\end{aligned} \tag{2.31}$$

where $\mathrm{d}^+_A a : \Omega^1(Y, \operatorname{Ad} P) \longrightarrow \Omega^2_+(Y, \operatorname{Ad} P)$ denotes the composition of the differential $\mathrm{d}_A$ and the projection of the 2-form part onto $\Omega^+(Y)$. This operator governs the infinitesimal deformations of ASD instantons, as stated in the following proposition:

**Proposition 2.32** (Proposition 4.2.23 in [DK90]). *For any connection $A$ on $P$ let*

$$\Gamma_A := \{u \in \mathscr{G} : u(A) = A\}.$$

*If $A$ is an ASD instanton, then a neighbourhood of $[A]$ in $M$ is modelled on a neighbourhood of $0$ of the quotient $f^{-1}(0)/\Gamma_A$ where*

$$f : \operatorname{Ker} \delta_A \to \operatorname{CoKer} \mathrm{d}_A^+$$

*is a $\Gamma_A$-equivariant map.*

We will also make use of the following Weitzenböck formula for the operator $\delta_A$:

**Proposition 2.33** (Equation 6.2.5 in [FU91]). *Let $P$ be a principal bundle over $Y$, and $A$ a connection on $P$ and $\widetilde{\delta}_A = \mathrm{d}_A^* \oplus \sqrt{2}\,\mathrm{d}_A^+ : \Omega^1(Y, \operatorname{Ad} P) \to \Omega^0(Y, \operatorname{Ad} P) \oplus \Omega^2_+(Y, \operatorname{Ad} P)$. Then*

$$\widetilde{\delta}_A^* \widetilde{\delta}_A a = \nabla_A^* \nabla_A a + \{\mathrm{Ric}, a\} + \{F_A^-, a\},$$

*where $F_A^-$ denotes the projection of the $2$-form part of $F_A$ onto $\Omega^-(Y)$, and $\{\cdot, \cdot\}$ denote universal bilinear forms.*

We then have the following index formula for $\delta_A$:

**Proposition 2.34** (Equation 4.2.22 in [DK90]). *Let $P$ be a bundle with structure group $\mathrm{SO}(3)$ over $Y$, and $A$ an ASD instanton. Then*

$$\operatorname{ind} \delta_A = -2p_1(E) - 3(1 - b_1(Y) + b_+(Y)).$$

One last result to mention is the classification of $\mathrm{SO}(3)$-bundles and $\mathrm{SU}(2)$-bundles. It will be mentioned in passing in Sections 2.5 and 4.6 but is not used in an essential way anywhere.

**Theorem 2.35** (Theorem 1 in [DW59] and Theorem E.8 in [FU91]). *Let $P, Q$ be $\mathrm{SO}(3)$-bundles over a compact $4$-manifold $Y$. Then $P$ and $Q$ are isomorphic if and only if $p_1(P) = p_1(Q)$ and $w_2(P) = w_2(Q)$.*

**Theorem 2.36** (Theorem E.5 in [FU91]). *Let $P, Q$ be* $\mathrm{SU}(2)$*-bundles over a compact* $4$*-manifold* $Y$*. Then $P$ and $Q$ are isomorphic if and only if* $c_2(P) = c_2(Q)$.

### 2.4.2 On ALE Manifolds

Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup and let $X$ be an ALE 4-manifold asymptotic to $\mathbb{C}^2/\Gamma$. Even though $X$ is non-compact, some of the results from gauge theory on compact manifolds carry over to this setting. First, we explain a correspondence between gauge equivalence classes of connections on $X$ and on its one point compactification $\hat{X} = X \cup \{\infty\}$. The following proposition explains the orbifold structure on $\hat{X}$:

**Proposition 2.37** (p.687 in [Kro89b] and Proposition 2.36 in [Wal13b]). *Let $(X, g)$ be an ALE manifold asymptotic to $\mathbb{C}^2/\Gamma$ by means of a map $\pi : X \to \mathbb{C}^2/\Gamma$ in the sense of Definition 2.8, and let $\hat{X} = X \cup \{\infty\}$ be the one point compactification of $X$.*

1. *The topological space $\hat{X}$ is an orbifold and there exist a neighbourhood $V$ of $\infty$ and an orbifold chart $f : B^4/\Gamma \to V$, where $B^4$ is the unit ball in $\mathbb{R}^4$.*
2. *The orbifold $\hat{X}$ carries an orbifold metric $\hat{g}$ of regularity $C^{3,\alpha}$ for any $\alpha \in (0, 1)$ such that the restriction of $\hat{g}$ to $X \subset \hat{X}$ is conformally equivalent to $g$.*

*Proof sketch.*

1. Fix an orientation reversing linear isometry $\sigma$ of $\mathbb{R}^4$. Let $\Gamma$ act on $B^4 \subset \mathbb{R}^4$ by $(g, x) \mapsto \sigma^{-1}(g \cdot \sigma(x))$ and define

$$f : B^4/\Gamma \to \hat{X}$$
$$x \mapsto \begin{cases} \infty & \text{if } x = 0 \\ \pi^{-1}(\sigma(x)/|x|^2) & \text{otherwise.} \end{cases} \tag{2.38}$$

2. The metric $\hat{g} := (1 + |\pi|^2)^{-2} g$ on $X$ is shown in [Kro89b, p.687] to extend to $\hat{X}$ as an orbifold metric with regularity $C^{3,\alpha}$ and is by definition conformally equivalent to $g$.

□

Let $G$ be a compact connected Lie group with a faithful representation $G \to \mathrm{GL}(V)$. Let $\hat{P}$ be an orbifold $G$-bundle over $\hat{X}$ and denote its restriction to $X$ by $P$, i.e. $P = \hat{P}|_X$. That is, $\hat{P}$ restricted to $V \simeq B^4/\Gamma$ from Proposition 2.37 is the trivial bundle $B^4 \times G$ together with a fixed lift of the action of $\Gamma$ on $B^4$ to $B^4 \times G$. Over the point $0 \in B^4$, this defines a homomorphism $\rho : \Gamma \to G$. The following proposition states that this homomorphism essentially characterises the orbifold bundle over $B^4$ completely.

**Proposition 2.39.** *There exists a trivialisation $\kappa : \hat{P}|_{B^4} \to B^4 \times G$ such that $\Gamma$ acts through left multiplication by $\rho$:*

$$\gamma \cdot \kappa^{-1}(b, g) = \kappa^{-1}(\gamma \cdot b, \rho(\gamma) g) \textit{ for } \gamma \in \Gamma, (b, g) \in B^4 \times G. \tag{2.40}$$

*Proof.* The lift of the action of $\Gamma$ to $B^4 \times G$ can be viewed as an element $w \in C^\infty(B^4, \mathrm{Hom}(\Gamma, G))$ via $\gamma \cdot (b, g) = (\gamma \cdot b, w(b)(\gamma) \cdot g)$. The space $B^4$ is connected, so by Corollary A.12 the conjugacy class of $w$ does not change over $B^4$. That is, there exists $\sigma \in C^\infty(B^4, G)$ such that $l_\sigma r_{\sigma^{-1}} w \in C^\infty(B^4, \mathrm{Hom}(\Gamma, G))$ is constant and $l_\sigma r_{\sigma^{-1}} w(0) = \rho$. Thus $\sigma$ defines a trivialisation of $B^4 \times G$ in which $\Gamma$ acts through left multiplication via $\rho$. □

Because of Proposition 2.39 we can fix a trivialisation of $\hat{P}$ over $B^4$ such that $\Gamma$ acts through left multiplication by $\rho$. Then denote by $A_0$ any extension of the product connection with respect to this trivialisation to all of $\hat{P}$. Different choices of extension will give rise to the very same spaces in Eq. (2.43). We identify $[R, \infty) \times S^3/\Gamma \simeq X \setminus K$ for some $R > 0$ big enough and a compact set $K \subset X$. Then the monodromy representation of $A_0$ restricted to $\{t\} \times S^3/\Gamma$, say $h : \pi_1(\{t\} \times S^3/\Gamma) \to G$, satisfies

$$h = \rho \tag{2.41}$$

under the canonical identification $\Gamma \simeq \pi_1(\{t\} \times S^3/\Gamma)$. Extend the projection onto the first component $X \setminus K \simeq [R, \infty) \times S^3 \longrightarrow [R, \infty)$ to a smooth positive function $r$ on all of $X$. For a non-negative integer $l$, a weight $\delta \in \mathbb{R}$, and $p \geq 1$ define the weighted Sobolev norm on the $k$-forms with values in the adjoint bundle with compact support $\Omega^k_0(\mathrm{Ad}\, P)$ via

$$||\alpha||_{L^p_{l,\delta}} = \sum_{j=0}^{l} \left( \int_X |\nabla^j_{A_0} \alpha|^p r^{-(\delta-j)p-4} \,\mathrm{d}V \right)^{1/p}, \tag{2.42}$$

and denote by $L^p_{l,\delta}(\Lambda^k(\mathrm{Ad}\, P))$ the completion of $\Omega^k_0(\mathrm{Ad}\, P)$ with respect to the norm $||\alpha||_{L^p_{l,\delta}}$.

As before, set $E = P \times_G V$ and for $l \geq 3$ define

$$\begin{aligned}
\mathscr{A}^{l,\delta} &= \{A_0 + \alpha : \alpha \in L^2_{l,\delta}(\Lambda^1(\mathrm{Ad}\, P))\}, \\
\mathscr{G}_0^{l+1,\delta+1} &= \{s \in L^2_{l+1,\mathrm{loc}}(\Lambda^0(\mathrm{End}(E)) : s(x) \in G \text{ for all } x \in G, ||s - \mathrm{Id}||_{L^2_{l+1,\delta+1}} < \infty\}, \\
G_\rho &= \{s \in G : s\rho s^{-1} = \rho\}, \\
\mathscr{G}^{l+1,\delta+1} &= \{s \in L^2_{l+1,\mathrm{loc}}(\Lambda^0(\mathrm{End}(E)) : s(x) \in G \text{ for all } x \in G, \\
&\qquad\qquad\qquad\qquad ||s - s_\infty||_{L^2_{l+1,\delta+1}} < \infty \text{ for some } s_\infty \in G_\rho\}.
\end{aligned} \tag{2.43}$$

In the definition of $\mathscr{G}^{l+1,\delta+1}$ we regarded $s_\infty \in G_\rho$ as an element in $C^\infty(\Lambda^0(\mathrm{End}(E))$ as follows: consider $\hat{P}$ over $B^4$ defined by the orbifold chart around $\infty$. Using the trivialisation from Proposition 2.39, this canonically defines a gauge transformation over $B^4$. (It is the same to say that we obtain a gauge transformation by parallel transport with respect to $A_0$.) This gauge transformation is $\Gamma$-equivariant by definition of $G_\rho$ and Proposition 2.39. We then extend it arbitrarily on the rest of $\hat{X}$ to an element in $C^\infty(\Lambda^0(\mathrm{End}(E))$. The choice of the extension does not matter for the condition $||s - s_\infty||_{L^2_{l+1,\delta+1}} < \infty$.

The gauge groups $\mathscr{G}_0^{l+1,\delta+1}$ and $\mathscr{G}^{l+1,\delta+1}$ both act on $\mathscr{A}^{l,\delta}$, and the quotient spaces $\mathscr{A}^{l,\delta}/\mathscr{G}_0^{l+1,\delta+1}$ and $\mathscr{A}^{l,\delta}/\mathscr{G}^{l+1,\delta+1}$ are called the moduli space of framed connections and the moduli space of unframed connections, respectively. We can restrict to anti-self-dual connections:

$$\mathscr{A}^{l,\delta}_{\mathrm{asd}} = \{A \in \mathscr{A}^{l,\delta} : A \text{ is anti-self-dual}\}$$

and obtain the *moduli space of framed ASD connections* $M^{l,\delta} := \mathscr{A}^{l,\delta}_{asd}/\mathscr{G}_0^{l+1,\delta+1}$ and the *moduli* *space of ASD connections* $\mathscr{A}^{l,\delta}_{asd}/\mathscr{G}^{l+1,\delta+1}$.

The four quotient spaces $\mathscr{A}^{l,\delta}/\mathscr{G}^{l+1,\delta+1}_0$, $\mathscr{A}^{l,\delta}/\mathscr{G}^{l+1,\delta+1}$, $M^{l,\delta}$, and $\mathscr{A}^{l,\delta}_{\mathrm{asd}}/\mathscr{G}^{l+1,\delta+1}$ are topological spaces. For $M^{l,\delta}$ we will observe explicitly (cf. Theorem 2.49) that it is metrisable and therefore Hausdorff, and the same argument works for the other three quotient spaces, cf. [DK90, Lemma 4.2.4].

Moving on to the orbifold, we define:

*Definition* 2.44. For $l \geq 3$ let

$$\begin{aligned}
\mathscr{A}^{l,\mathrm{orb}}_{\mathrm{asd}} &= \{A_0 + \alpha : \alpha \in L^2_l(\Lambda^1(\mathrm{Ad}\,\hat{P}))\},\\
\mathscr{G}^{l+1,\mathrm{orb}} &= \{s \in L^2_{l+1}(\Lambda^0(\mathrm{End}\,V)) : s(x) \in G \text{ for all } x \in \hat{X}, s(\infty) \in G_\rho\},\\
\mathscr{G}^{l+1,\mathrm{orb}}_0 &= \{s \in \mathscr{G}^{l+1,\mathrm{orb}} : s(\infty) = \mathrm{Id}\}.
\end{aligned}$$

Then $\mathscr{G}^{l+1,\mathrm{orb}}$ and $\mathscr{G}^{l+1,\mathrm{orb}}_0$ both act on $\mathscr{A}^{l,\mathrm{orb}}_{\mathrm{asd}}$ and we can form the quotient spaces $\mathscr{A}^{l,\mathrm{orb}}_{\mathrm{asd}}/\mathscr{G}^{l+1,\mathrm{orb}}$ and $M^{l,\mathrm{orb}} = \mathscr{A}^{l,\mathrm{orb}}_{\mathrm{asd}}/\mathscr{G}^{l+1,\mathrm{orb}}_0$. Here, $M^{l,\mathrm{orb}}$ is called the *moduli space of framed ASD connections* on $\hat{X}$.

We also have the following analogue of Proposition 2.30.

**Proposition 2.45.** *For* $3 \leq l_1 < l_2$*, the inclusion maps*

$$M^{l_1,orb} \hookrightarrow M^{l_2,orb}, \qquad\qquad M^{l_1,-2} \hookrightarrow M^{l_2,-2}$$

*are homeomorphisms.*

The proof of Proposition 2.45 works the same as in the compact case, i.e. the proof of Proposition 2.30 given in [DK90, Proposition 4.2.16]. The only difference is that in the non-compact case, i.e. for the claim $M^{l_1,-2} \hookrightarrow M^{l_2,-2}$, one has to take the weighted Sobolev norms from Eq. (2.42). These have their own versions of the Sobolev embedding theorem and, if the weight is non-positive, the multiplication theorem for Sobolev norms also holds. These properties of weighted Sobolev norms are proved in [Pac13, Corollary 6.8].

**Proposition 2.46.** *For any* $A \in \mathscr{A}^{l,-2}_{\mathrm{asd}}$ *there exists a connection* $\hat{A} \in \mathscr{A}(\hat{P})$ *satisfying* $\hat{A}|_P = A$*.*

*Proof.* Corollary A.17 gives a bundle $P'$ over $\hat{X}$ with connection $A'$ together with an injective bundle homomorphism $\xi : P \to P'$. After fixing a trivialisation of $\hat{P}$ around $\infty$, this canonically defines an isomorphism of orbifold $G$-bundles $h : \hat{P} \to P'$, and $\hat{A} := h^*(A')$ satisfies $\hat{A}|_P = A$. $\square$

*Definition* 2.47. Define the map

$$\Psi : M^{3,-2} \to M^{3,\mathrm{orb}}$$

as follows: for $[A_0 + a] \in M^{3,-2}$ let $\hat{A} \in \mathscr{A}(\hat{P})$ be the induced connection from Proposition 2.46 and set $\Psi([A_0 + a]) := [\hat{A}]$.

**Proposition 2.48.** *The function $\Psi$ from Definition 2.47 is bijective.*

*Proof.* $\Psi$ **is injective**: let $[A_0 + a], [A_0 + \widetilde{a}] \in M^{3,-2}$ such that $\Psi([A_0 + a]) = [\hat{A}]$ as well as $\Psi([A_0 + \widetilde{a}]) = [\hat{A}']$. If $[\hat{A}] = [\hat{A}']$, then $\hat{A}' = s\hat{A}$ for some $s \in \mathscr{G}_0^{4,\mathrm{orb}}$. We have $s(\infty) = \mathrm{Id}$, so $(s - \mathrm{Id}) = O(|x|)$ and $\nabla^k_{A_0}(s - \mathrm{Id}) = O(1)$ for $k \in \{1, 2, 3, 4\}$. Here, $\nabla^k_{A_0}$ includes terms containing the Levi-Civita connection for the orbifold metric $\hat{g}$ on $\hat{X}$ for $k > 1$, and $|x|$ denotes the distance from $\infty \in \hat{X}$ in this metric. In particular, $\nabla^k_{A_0}(s - \mathrm{Id}) = O(|x|^{1-k})$. We have

$$\left|\nabla^k_{A_0}(s - \mathrm{Id})\right|_g = (1 + r^2)^{-k} \left|\nabla^k_{A_0}(s - \mathrm{Id})\right|_{\hat{g}} = O(r^{-2k} |x|^{1-k}) = O(r^{-1-k}),$$

where $g$ denotes the ALE metric, in the first step we used the definition of $\hat{g}$ from the proof of Proposition 2.37 and the fact that we are measuring a tensor with $k$ covariant indices and 0 contravariant indices. Thus, $s \in \mathscr{G}_0^{4,-1}$. Therefore, $[A_0 + a] = [A_0 + \widetilde{a}]$ as elements in $M^{3,-2}$, which shows the claim.

$\Psi$ **is surjective**: Let $[A_0 + a] \in M^{3,\mathrm{orb}}$, i.e. $A_0 + a \in \mathscr{A}^{3,\mathrm{orb}}_{\mathrm{asd}}$. Similar to the previous point we find that $\nabla^k_{A_0} a = O(r^{-2-k})$. By construction $\Psi([(A_0 + a)|_X]) = [A_0 + a]$, which proves the claim. $\square$

Because of Proposition 2.45 we will drop the regularity and decay from the notation of our moduli spaces most of the time. That is, we will often write $M$ for $M^{l,\delta}$ with any $l \geq 3$ and $\delta = -2$. Likewise for $\mathscr{A}, \mathscr{G}, \mathscr{G}_0, \mathscr{A}^{\mathrm{orb}}, M^{\mathrm{orb}}, \mathscr{G}^{\mathrm{orb}}$, and $\mathscr{G}_0^{\mathrm{orb}}$.

The important results about the local structure of $M$ are the following:

**Theorem 2.49** (Theorem 2.4 and Proposition 5.1 in [Nak90]). *$M$ is a nonsingular smooth manifold and for $[A] \in M$ its tangent space is isomorphic to*

$$H^1_{A,-2} := \{\alpha \in L^2_{l,-2}(\Lambda^1(\operatorname{Ad} P)) : \delta_A(\alpha) = 0\}.$$

For the linear operator $\delta_A$ we have the following analytic result:

**Proposition 2.50** (Proposition 5.10 in [Wal13a]). *Let $A \in \mathscr{A}(E)$ be a finite energy ASD instanton on $E$. Then the following holds:*

1. *If $a \in \operatorname{Ker} \delta_A$ decays to zero at infinity, i.e., $\lim_{r\to\infty} \sup_{\rho(x)=r} |a|(x) = 0$, then $\nabla^k_A a = O(|\pi|^{-3-k})$ for all $k \geq 0$.*
2. *If $(\xi, \omega) \in \operatorname{Ker} \delta^*_A$ decays to zero at infinity, then $(\xi, \omega) = 0$.*

The Hyperkähler triple of $X$ acts on the 1-form part of $\Omega^1(\operatorname{Ad} P)$. It is checked in [Ito88, Section 4] together with [Ito85, Proposition 2.4] that this action restricts to $H^1_{A,-2}$ for all $[A] \in M$. We thus have a triple of complex structures on $M$. The following theorem states that this defines a Hyperkähler structure with respect to the standard metric on $M$:

**Theorem 2.51** (Theorem 2.6 and Proposition 5.1 in [Nak90]). *The metric $g_M$ defined by*

$$g_M(\alpha, \beta) = \int_X g(\alpha, \beta) \operatorname{vol}_X \qquad \text{for } \alpha, \beta \in H^1_{A,-2}$$

*and the Hyperkähler triple defined by acting with the Hyperkähler triple of $X$ on the 1-form part of $\Omega^1(\operatorname{Ad} P)$ is well-defined on $M$ and defines a Hyperkähler structure on $M$.*

**Theorem 2.52** (Theorem 2.47 in [Wal13b]). *Let $\rho : \Gamma \to G$ be a homomorphism, $A_0$ a connection on a bundle $P$ that is flat at infinity as in Proposition 2.39 whose holonomy representation is equal to $\rho$ in the sense of Eq. (2.41). Let $\delta \in (-3, -1)$ and $A = A_0 + \alpha$ for some $\alpha \in L^2_{1,\delta}(\Lambda^1(\operatorname{Ad} P))$.*

*Then the $L^2$ index of $\delta_A$, defined as*

$$\dim\{a \in L^2(\Lambda^1(\operatorname{Ad} P)) \cap C^\infty(\Lambda^1(\operatorname{Ad} P)) : \delta_A(a) = 0\}$$
$$- \dim\{\underline{a} \in L^2(\Lambda^0 \oplus \Lambda^2_+(\operatorname{Ad} P)) \cap C^\infty(\Lambda^0 \oplus \Lambda^2_+(\operatorname{Ad} P)) : \delta_A^*(\underline{a}) = 0\},$$

*is given by*

$$\operatorname{ind} \delta_A = -2 \int_X p_1(\operatorname{Ad} P) + \frac{2}{|\Gamma|} \sum_{g \in \Gamma \setminus \{e\}} \frac{\chi_{\mathfrak{g}}(g) - \dim \mathfrak{g}}{2 - \operatorname{tr} g}. \tag{2.53}$$

*Here $p_1(\operatorname{Ad} P)$ is the Chern-Weil representative of the first Pontrjagin class of $P$ and $\chi_{\mathfrak{g}}$ is the character of $g$ acting on $\mathfrak{g}$, the Lie algebra associated with $G$, via $\rho$, and $\operatorname{tr} g$ is the trace of $g$ acting on $\mathfrak{g}$. Moreover, if $A$ is an ASD instanton, then $\operatorname{ind} \delta_A = \dim \operatorname{Ker} \delta_A = \dim M$.*

Here come two examples of anti-self-dual instantons on ALE spaces. First, recall the construction of $X_{\mathrm{EH}}$ as a Hyperkähler quotient and the Hyperkähler moment map $\mu$ from Eq. (2.15). Using this notation, we have the following result from [GN92].

**Proposition 2.54** (Section 2 in [GN92]). *The $\mathrm{U}(1)$-bundle $\mathcal{R} := \mu^{-1}(i/2) \to X_{EH} = \mu^{-1}(i/2)/\mathrm{U}(1)$ admits a non-flat finite energy ASD instanton $A$ asymptotic to the representation $\rho : \mathbb{Z}_2 \to \mathrm{U}(1)$ determined by $\rho(-1) = -1$ in the sense of Eq. (2.41).*

An additional property of $\mathcal{R}$ that we will need later is the following:

**Proposition 2.55.** *There exists a lift of the action of the holomorphic isometry group $\mathrm{U}(2)/\{\pm 1\}$ of $X_{EH}$ to $\mathcal{R}$.*

*Proof.* We have seen in the construction of $X_{\mathrm{EH}}$ as a Hyperkähler quotient before Proposition 2.16 that the holomorphic isometry group $\mathrm{U}(2)/\{\pm 1\}$ is realised as an action of $\mathrm{U}(2)/\{\pm 1\}$ on $\mu^{-1}(i/2)$ that commutes with the action of $\mathrm{U}(1)$ on $\mu^{-1}(i/2)$. The action of $\mathrm{U}(2)/\{\pm 1\}$ on $\mu^{-1}(i/2)$ is the desired lift of the action of $\mathrm{U}(2)/\{\pm 1\}$ on $X_{\mathrm{EH}}$. □

*Remark* 2.56. We can apply Theorem 2.52 to the $\mathrm{U}(1)$-bundle over $X_{\mathrm{EH}}$ defined before to find that it is rigid. As $\operatorname{Ad} \mathcal{R}$ has rank 1, we have that $p_1(\operatorname{Ad} \mathcal{R}) = c_2(\operatorname{Ad} \mathcal{R}^{\mathbb{C}}) = 0$, and plugging this into the index formula from Theorem 2.52 proves the claim.

*Remark* 2.57. On simply connected compact manifolds it is the case that any U(1)-bundle admits an ASD-instanton that is unique up to the action of the gauge group. This is a consequence of the Hodge theorem. On non-compact manifolds a variation of the Hodge theorem for $L^2$-forms holds, see [Loc87, Example 0.15], and can be used to give an alternative proof of Remark 2.56 without the use of the index formula.

Here is a non-rigid example:

*Example* 2.58 (Chapter II in [Ati78]). Consider the BPST instantons from [BPST75] on $\mathbb{R}^4$. On the trivial SU(2)-bundle $P$ over $\mathbb{R}^4$ define a connection via

$$A = \frac{1}{1+|x|^2}(\theta_1 i + \theta_2 j + \theta_3 k)$$

where $i, j, k$ is the standard basis for the space of unit quaternions $\mathfrak{sp}(1) \simeq \mathfrak{su}(2)$ and

$$\begin{aligned}
\theta_1 &= x_1 \,\mathrm{d}x_2 - x_2 \,\mathrm{d}x_1 - x_3 \,\mathrm{d}x_4 + x_4 \,\mathrm{d}x_3,\\
\theta_2 &= x_1 \,\mathrm{d}x_3 - x_3 \,\mathrm{d}x_1 - x_4 \,\mathrm{d}x_2 + x_2 \,\mathrm{d}x_4,\\
\theta_3 &= x_1 \,\mathrm{d}x_4 - x_4 \,\mathrm{d}x_1 - x_2 \,\mathrm{d}x_3 + x_3 \,\mathrm{d}x_2.
\end{aligned}$$

Then $A$ has curvature

$$F_A = \left(\frac{1}{1+|x|^2}\right)^2 (\mathrm{d}\theta_1 i + \mathrm{d}\theta_2 j + \mathrm{d}\theta_3 k)$$

and a computations shows that $A$ is an ASD-instanton. The Killing form on $\mathfrak{sp}(1)$ is given by

$$\langle u_1, u_2 \rangle = -8 \operatorname{Re}(u_1 u_2) \text{ for } u_1, u_2 \in \mathfrak{sp}(1)$$

which gives

$$\begin{aligned}
\int_{\mathbb{R}^4} p_1(\operatorname{Ad} P) &= -\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \langle F_A, F_A \rangle \operatorname{vol}_{\mathbb{R}^4}\\
&= -\frac{24}{\pi^2}\int_{\mathbb{R}^4} \left(\frac{1}{1+|x|^2}\right)^4 \operatorname{vol}_{\mathbb{R}^4}\\
&= -48 \int_0^\infty \left(\frac{1}{1+r^2}\right)^4 \mathrm{d}r\\
&= -4.
\end{aligned}$$

Thus, by Theorem 2.52, $A$ lives in an 8-dimensional moduli space of framed ASD-instantons. This moduli space is given by the following connections: for any $y \in \mathbb{R}^4$ and $\lambda \in \mathbb{R}^+$ we get another ASD instanton by translating by $y$ and dilating with $\lambda$. One checks that different choices of $y$ and $\lambda$ give rise to connections which are not gauge equivalent. The connection $A$ is irreducible, so its orbit under the action of $\mathscr{G}/\mathscr{G}_0 = G_\rho = G$ is isomorphic to $G_\rho/C(G) = \mathrm{SU}(2)/\{\pm 1\} = \mathrm{SO}(3)$ by [Nak90, p. 275]. The framed moduli space $M$ is thus diffeomorphic to $\mathbb{R}^4 \times \mathbb{R}^+ \times \mathrm{SO}(3)$.

Before ending the section we will state two results about universal bundles that will be needed later. The proof of the following proposition is based on the proof of [DK90, Proposition 5.2.17].

**Proposition 2.59.** *There exist*

- *a $G$-bundle $\widetilde{\mathbb{P}}$ over $M \times \hat{X}$ with a natural action of $G_\rho \simeq \mathscr{G}/\mathscr{G}_0$ on $\widetilde{\mathbb{P}}$ covering the action of $G_\rho$ on $M$,*
- *a connection $\widetilde{\mathbb{A}} \in \mathscr{A}(\widetilde{\mathbb{P}})$ that is invariant under the action of $G_\rho \simeq \mathscr{G}/\mathscr{G}_0$, and*
- *for each choice of $\phi \in \mathrm{Iso}_\Gamma(G, P_\infty)$ a canonical isomorphism of $G$-bundles with $\Gamma$ left action $\underline{\phi} : \widetilde{\mathbb{P}}|_{M\times\{\infty\}} \to G \times M$*

*satisfying:*

- *for any element $[A] \in M$ there exists an isomorphism $\widetilde{\mathbb{P}}|_{\{[A]\}\times\hat{X}} \simeq \hat{P}$ such that under this isomorphism $\widetilde{\mathbb{A}}|_{\{[A]\}\times X}$ and $A$ agree up to the action of $\mathscr{G}_0$.*
- *if we decompose the curvature of $\widetilde{\mathbb{A}}$ over $M\times X$ according to the bi-grading on $\Lambda^*T^*(M\times X)$ induced by $T^*(M\times X) = \pi_1^*T^*M \oplus \pi_2^*T^*X$, then its components satisfy the following:*
  - *$F^{1,1}_{\widetilde{\mathbb{A}}} \in \Gamma(\mathrm{Hom}(\pi_1^*T^*M, \pi_2^*T^*X \otimes \mathrm{Ad}\,P))$ at $([A], x)$ is the evaluation of $a \in T_{[A]}M$ at $x$,*
  - *$F^{0,2}_{\widetilde{\mathbb{A}}} \in \Gamma(\pi_2^*\Lambda^-(X)^* \otimes \mathrm{Ad}\,P)$, where $\Lambda^-$ is defined using the ALE metric on $X$,*
- *$\underline{\phi}^*A_{product} = \widetilde{\mathbb{A}}|_{M\times\{\infty\}}$, where $A_{product} \in \mathscr{A}(G \times M)$ denotes the product connection.*

The proof makes use of the following lemma. Here, the data $\Gamma$, $\hat{E}$, $\hat{Y}$ can be taken to be infinite-dimensional, which is the version of the statement that we use later.

**Lemma 2.60** (Equation 5.2.16 in [DK90]). *Let $Y$ and $\hat{Y}$ be smooth manifolds, $\hat{E} \to \hat{Y}$ vector bundle, and suppose a group $\Gamma$ acts smoothly on $\hat{E}$, covering a free action on $\hat{Y}$. Let $E = \hat{E}/\Gamma \to Y = \hat{Y}/\Gamma$ be the quotient. The data of*

*(i) a connection $\hat{\nabla}$ in $\hat{E}$ which is invariant under $\Gamma$,*

*(ii) a connection in the $\Gamma$-bundle $p : \hat{Y} \to Y$, determined by a horizontal distribution $H$*

*define a connection $\nabla$ on $E$ via*

$$(\nabla_U s)^\wedge = \hat{\nabla}_{\hat{U}} \hat{s}, \tag{2.61}$$

*in which $s$ is a section of $E$ corresponding to a local invariant section $\hat{s} : \hat{Y} \to \hat{E}$ and $\hat{U}$ is a horizontal lift of $U$ with respect to $H$. This definition is independent of the choice of lift and the curvature of $\nabla$ satisfies*

$$F(\nabla)(U,V)^\wedge = F(\hat{\nabla})(\hat{U},\hat{V}) - \Phi \circ (\Theta(U,V)), \tag{2.62}$$

*where $U, V \in T_{[y]}Y$, $\hat{U}, \hat{V} \in T_{[y]}\hat{Y}$ are horizontal lifts with respect to $H$, $\Phi : \hat{Y} \times_\Gamma \mathrm{Lie}(\Gamma) \to \mathrm{End}\,\hat{E}$ is a linear map, and $\Theta$ is the curvature of $H$.*

*Proof of Proposition 2.59.* Let $E$ be the vector bundle associated to $\hat{P}$ by means of a faithful representation of $G$. Then we will apply Lemma 2.60 in the case $\hat{Y} = \mathscr{A}^{\mathrm{orb}}_{\mathrm{asd}} \times \hat{X}$, $\Gamma = \mathscr{G}^{\mathrm{orb}}_0$. Let $\hat{E} = \pi_2^* E$, where $\pi_2 : \mathscr{A}^{\mathrm{orb}}_{\mathrm{asd}} \times \hat{X} \to \hat{X}$ is the projection onto the second factor. The orbifold gauge group $\mathscr{G}^{\mathrm{orb}}_0$ then acts through pullback on $\hat{E}$.

$\hat{E}$ carries a tautological connection $\hat{\nabla}$ characterised by the properties that $\hat{\nabla}|_{\mathscr{A}^{\mathrm{orb}}_{\mathrm{asd}} \times \{x\}}$ is trivial and $\hat{\nabla}|_{\{A\}\times\hat{X}} = A$ under the canonical isomorphism $\hat{E}|_{\{A\}\times\hat{X}} \simeq E$. The connection $\hat{\nabla}$ satisfies

$$\begin{aligned} F(\hat{\nabla})(u,v) &= F(A)(u,v), \\ F(\hat{\nabla})(a,v) &= \langle a,v\rangle, \\ F(\hat{\nabla})(a,b) &= 0 \end{aligned} \tag{2.63}$$

for $u,v \in T_x\hat{X}$ and $a,b \in T_A\mathscr{A}^{\mathrm{asd}}$.

We will now define horizontal subspaces in the bundle $\mathscr{A}^{\mathrm{orb}}_{\mathrm{asd}} \to M = \mathscr{A}^{\mathrm{orb}}_{\mathrm{asd}}/\mathscr{G}^{\mathrm{orb}}_0$. As a first step, we define the horizontal subspaces $H$ for the principal bundle $\mathscr{A}_{\mathrm{asd}} \to M = \mathscr{A}_{\mathrm{asd}}/\mathscr{G}_0$ as

$$H_A = \{a \in T_A\mathscr{A}_{\mathrm{asd}} = \Omega^1(X,\operatorname{Ad}P) : \mathrm{d}_A^* a = 0\}. \tag{2.64}$$

Here, the adjoint $\mathrm{d}_A^*$ is taken with respect to the ALE metric on $X$.

The $H_A$ are $\mathscr{G}_0$-invariant, i.e. for $s \in \mathscr{G}_0$ we have that $\mathrm{d}R_s(H_A) = H_{s^*A}$. To see this, let $a \in H_A$ and $u \in \Omega^0(X,\operatorname{Ad}P)$. Under the identification of $k$-forms taking values in the adjoint bundle with horizontal equivariant forms on $P$, we can view $a$ as an element in $\Omega^1(P,\mathfrak{g})$ and $u$ as an element in $\Omega^0(P,\mathfrak{g})$. Elements in $\mathscr{G}$ are in 1-to-1 correspondence with $G$-equivariant smooth maps $P \to G$, and we denote by $\sigma_s : P \to G$ the map corresponding to $s$. Then

$$\begin{aligned} \langle \mathrm{d}^*_{s^*A}(\mathrm{d}R_s(a)), u\rangle &= \langle \mathrm{d}R_s(a), \mathrm{d}_{s^*A}u\rangle \\ &= \langle \operatorname{Ad}(\sigma_s^{-1})a, \mathrm{d}u\rangle + \langle \operatorname{Ad}(\sigma_s^{-1})a, [\operatorname{Ad}(\sigma_s^{-1})A, u]\rangle \\ &= \langle a, \mathrm{d}(\operatorname{Ad}(\sigma_s)u)\rangle + \langle a, \operatorname{Ad}(\sigma_s)[\operatorname{Ad}(\sigma_s^{-1})A, u]\rangle \\ &= \langle a, \mathrm{d}_A(\operatorname{Ad}(\sigma_s)u)\rangle \\ &= \langle \mathrm{d}_A^* a, \operatorname{Ad}(\sigma_s)u\rangle = 0, \end{aligned}$$

where we used that the Killing form is Ad-invariant in the third step, and we used the assumption $a \in H_A$ in the last step. As this holds for all $u \in \Omega^0(X,\operatorname{Ad}P)$, we have that $\mathrm{d}R_s(a) \in H_{s^*A}$. The fact that they are horizontal, i.e. a complement to the vertical space generated by the action of $\mathscr{G}_0$ on $\mathscr{A}^{\mathrm{asd}}$, is Theorem 2.49. We are now ready to write down the horizontal subspaces

$H'$ for the principal bundle $\mathscr{A}_{\mathrm{asd}}^{\mathrm{orb}} \to M = \mathscr{A}_{\mathrm{asd}}^{\mathrm{orb}}/\mathscr{G}_0^{\mathrm{orb}}$. Let

$$H'_A = \{a \in T_A\mathscr{A}_{\mathrm{asd}} = \Omega^1(\hat{X}, \operatorname{Ad}\hat{P}) : \mathrm{d}^*_{A|_X}(a|_X) = 0\}, \tag{2.65}$$

where again the Hodge star is taken with respect to the ALE metric. The subspaces $H'$ are right-invariant with the same proof as for $H$. To see that they are horizontal, note that they are not vertical, and satisfy

$$\operatorname{rank} H' = \operatorname{rank} H = \dim (M) = \dim \left(M^{\mathrm{orb}}\right).$$

The first step follows from the definitions of $H$ and $H'$, the second step is the fact that $H$ is horizontal, and the third step is Proposition 2.48. This shows that $H'$ defines a principal bundle connection.

By pullback, $H$ induces a connection on the principal bundle $\mathscr{A}_{\mathrm{asd}}^{\mathrm{orb}} \times X \to \mathscr{A}_{\mathrm{asd}}^{\mathrm{orb}}/\mathscr{G}_0^{\mathrm{orb}} \times X$ which is trivial in the $X$-direction. Therefore, its curvature $\Theta$ satisfies

$$\begin{aligned} \Theta(u, v) &= 0, \\ \Theta(a, v) &= 0 \end{aligned} \tag{2.66}$$

for $u, v \in T_xX$ and $a \in T_A\mathscr{A}_{\mathrm{asd}}^{\mathrm{orb}}$.

Lemma 2.60 then gives a connection $\nabla$ on $\underline{E} := \hat{E}/\mathscr{G}_0^{\mathrm{orb}}$. And Eqs. (2.62), (2.63) and (2.66) give for the curvature of $\nabla$ at the point $([A], x) \in M \times X$:

$$\begin{aligned} F(\hat{\nabla})(u, v) &= F(A)(u, v), \\ F(\hat{\nabla})(a, v) &= \langle a, v \rangle \end{aligned} \tag{2.67}$$

for $u, v \in T_xX$ and $a \in T_{[A]}M^{\mathrm{orb}} \simeq \operatorname{Ker} \delta_A \subset \Omega^1(\operatorname{Ad}\hat{P})$. Denote by $\widetilde{\mathbb{P}}$ a $G$-reduction of the bundle of frames of $\underline{E}$ and by $\widetilde{\mathbb{A}}$ the connection on $\widetilde{\mathbb{P}}$ induced by $\nabla$. The curvature of $\widetilde{\mathbb{A}}$ still satisfies the analogue of Eq. (2.67).

Last, any $\phi \in \operatorname{Iso}_\Gamma(G, P_\infty)$ pulls back to an isomorphism of vector bundles with $\Gamma$-left action

$\phi' : \hat{E}|_{\mathscr{A}^{\mathrm{orb}}_{\mathrm{asd}}\times\{\infty\}} \simeq E \times \mathscr{A}^{\mathrm{orb}}_{\mathrm{asd}}$. By definition of $\hat{\nabla}$, we have that

$$(\phi')^*(\nabla^{\mathrm{product}}) = \hat{\nabla}|_{\mathscr{A}^{\mathrm{orb}}_{\mathrm{asd}}\times\{\infty\}}. \tag{2.68}$$

Using that $\phi'$ is $\mathscr{G}_0$-equivariant and changing to the bundle of frames, we get an isomorphism $\underline{\phi} : \mathbb{P}|_{M^{\mathrm{orb}}\times\{\infty\}} \to G \times M^{\mathrm{orb}}$ of the quotient. Lastly, because of Eq. (2.68), we have that $\underline{\phi}^* A_{\mathrm{product}} = \widetilde{\mathbb{A}}|_{M\times\{\infty\}}$. □

By Proposition A.1, the group of holomorphic isometries acting on $X_{\mathrm{EH}}$ is $\mathrm{U}(2)/\{\pm 1\}$. This induces a non-effective action of $\mathrm{U}(2)$ on $\hat{X}_{\mathrm{EH}}$ by demanding that each group element fixes $\infty \in \hat{X}_{\mathrm{EH}}$. Then $\mathrm{U}(2)$ acts from the left on $M$ (and equally $M^{\mathrm{orb}}$) as follows: $\mathrm{U}(2)$ is connected, so $(u^{-1})^*E$ and $E$ are homotopic bundles and in particular isomorphic. Different choices of isomorphism give rise to gauge equivalent connections, so $[(u^{-1})^*A] \in M$ is well-defined.

Later on (cf. Definition 4.9) we will need the following assumption:

*Assumption* 2.69. The action of $\mathrm{U}(2)$ on $M \times \hat{X}_{\mathrm{EH}}$ can be lifted to an action on $\widetilde{\mathbb{P}}$ that preserves $\widetilde{\mathbb{A}}$.

In the examples constructed in Section 4.6 this assumption will be satisfied because of the following proposition:

**Proposition 2.70.** *Let $\widetilde{\mathbb{P}} \to M \times \hat{X}_{EH}$ be the tautological bundle with tautological connection $\widetilde{\mathbb{A}}$ from Proposition 2.59.*

*If the action of $\mathrm{U}(2)$ on $\hat{X}_{EH}$ can be lifted to an action on $\hat{P}$, then the action of $\mathrm{U}(2)$ on $M \times \hat{X}_{EH}$ can be lifted to an action on $\widetilde{\mathbb{P}}$. If it exists, this lift can be chosen to preserve $\widetilde{\mathbb{A}}$.*

*Proof.* First, assume that the action of $\mathrm{U}(2)$ on $\hat{X}_{\mathrm{EH}}$ can be lifted to an action on $\hat{P}$. This is equivalent to saying that for all $g \in G$ there exists a bundle isomorphism $\xi_g : \hat{P} \to \hat{P}$ covering $g : \hat{X}_{\mathrm{EH}} \to \hat{X}_{\mathrm{EH}}$. Recall that $\widetilde{\mathbb{P}} \simeq \pi_2^*\hat{P}/\mathscr{G}_0^{\mathrm{orb}}$, where $\pi_2 : \mathscr{A}^{\mathrm{orb}}_{\mathrm{asd}} \times \hat{X}_{\mathrm{EH}} \to \hat{X}_{\mathrm{EH}}$ is the projection onto the second factor. Let $([A], x) \in M \times \hat{X}_{\mathrm{EH}}$ and $[u] \in \widetilde{\mathbb{P}}_{([A],x)}$ where $u \in \left(\pi_2^*\hat{P}\right)_{(A,x)} \simeq \hat{P}_x$. We define $\kappa_g : \widetilde{\mathbb{P}} \to \widetilde{\mathbb{P}}$ covering $g : M \times \hat{X}_{\mathrm{EH}} \to M \times \hat{X}_{\mathrm{EH}}$ via $\kappa_g[u] := [\xi_g(u)]$. To check that this is well-defined, let $s \in \mathscr{G}_0^{\mathrm{orb}}$, and observe that $\kappa_g[su] = [(\xi_g s \xi_g^{-1})(\xi_g u)] = [\xi_g u]$.

It remains to show that this lift preserves $\widetilde{\mathbb{A}}$. First observe that the map

$$\hat{\kappa}_g : \pi_2^* P \longrightarrow \pi_2^* P$$
$$\left(\pi_2^* P\right)_{(A,x)} \ni u \longmapsto \xi_g(u) \in \left(\pi_2^* P\right)_{(\xi^*_{g^{-1}} A, gx)}$$

preserves the tautological connection $\hat{A}$, which is the principal bundle connection on $\pi_2^* P$ inducing $\hat{\nabla}$ on the associated vector bundle, because

$$(\hat{\kappa}_g^* \hat{A})|_{\{A\} \times X_{\mathrm{EH}}} = \hat{\kappa}_g^* (\hat{A}|_{\{\xi^*_{g^{-1}} A\} \times X_{\mathrm{EH}}}) = \xi_g^* \xi^*_{g^{-1}} A = A.$$

The action of $\mathrm{U}(2)$ on $\mathscr{A}^{\mathrm{orb}}_{\mathrm{asd}}$ also preserves the horizontal subspaces $H'$ from Eq. (2.65). By definition of $H'$ it suffices to check that the action of $\mathrm{U}(2)$ on $\mathscr{A}_{\mathrm{asd}}$ preserves the horizontal subspaces $H$ from Eq. (2.64). To this end, let $a \in H_A$, i.e. $d_A^* a = 0$. Then

$$\mathrm{d}^*_{\xi_{g^{-1}} A} \left( \xi^*_{g^{-1}} a \right) = * \, \mathrm{d}_{\xi_{g^{-1}} A} * \left( \xi^*_{g^{-1}} a \right) = * \, \mathrm{d}_{\xi_{g^{-1}} A} \left( \xi^*_{g^{-1}} (*a) \right) = * \xi^*_{g^{-1}} \left( \mathrm{d}_A (*a) \right) = \xi^*_{g^{-1}} (d_A^* a) = 0,$$

where in the second and fourth step we used that $g^{-1} : X_{\mathrm{EH}} \to X_{\mathrm{EH}}$ is an isometry, and in the third step we used that exterior differential and pullback commute. The connection $\widetilde{\mathbb{A}}$ was defined using the data of $\hat{A}$ and $H$ by means of Lemma 2.60. The action of $\mathrm{U}(2)$ preserves $\hat{A}$ and $H$ and therefore preserves $\widetilde{\mathbb{A}}$. □

## 2.5 Gauge Theory on Complex Vector Bundles

### 2.5.1 Hermite-Einstein Connections and Stable Bundles

Throughout the section, let $E$ be a complex vector bundle over a complex manifold $M$.

*Definition* 2.71. A bundle atlas of $E$ with holomorphic transition functions is called a *holomorphic structure on $E$*.

We will often use $\mathcal{E}$ to denote a complex vector bundle together with its holomorphic structure, and $E$ to denote the underlying complex vector bundle.

*Definition* 2.72. A map $\overline{\partial}_E : \Omega^0(M, E) \longrightarrow \Omega^{0,1}(M, E)$ that is $\mathbb{C}$-linear, satisfies the Leibniz rule

$\overline{\partial}_E(fs) = \overline{\partial}(f) \otimes s + f\overline{\partial}_E(s)$ for $f \in C^\infty(\mathbb{C})$ and $s \in \Gamma(E)$, and satisfies $\overline{\partial}_E^2 = 0$ is called a *Dolbeault Operator.*

Given a holomorphic structure, we get a Dolbeault operator by taking the canonical $\overline{\partial}$ in the trivialisations of the bundle atlas. The fact that transition functions are holomorphic guarantees that the resulting operator is well-defined on all of $M$, not just on one trivialisation. We have the following result that describes the relation between Dolbeault operators and connections:

*Definition* 2.73. For a Hermitian metric on $E$, denote by $\mathscr{A}^{1,1}$ the set of unitary connections with curvature of type $(1,1)$. Here, curvature of type $(1,1)$ means that in the decomposition of the curvature $F_A$ according to type, i.e. $F_A = F_A^{2,0} + F_A^{1,1} + F_A^{0,2}$, we have that $F_A^{2,0} = F_A^{0,2} = 0$. Denote by $\overline{\partial}_A = \mathrm{proj}_{\Omega^{0,1}} \circ d_A$ the *Dolbeault operator induced by $A$.*

**Proposition 2.74** (Proposition 4.2.14 in [Huy05]). *Let $\mathcal{E}$ be a holomorphic structure on $E$, fix a Hermitian metric on $E$ and let $\overline{\partial}_{\mathcal{E}}$ be a Dolbeault operator on $\mathcal{E}$. Then there exists a unique $A \in \mathscr{A}^{1,1}$ such that $\overline{\partial}_A = \overline{\partial}_{\mathcal{E}}$.*

The uniquely determined connection from Proposition 2.74 is called the *Chern connection.* One can also go the converse way: every $A \in \mathscr{A}^{1,1}$ is the Chern connection with respect to some holomorphic structure:

**Proposition 2.75** (Theorem 5.1 in [AHS78]). *Fix a Hermitian metric on $E$. For $A \in \mathscr{A}^{1,1}$, there exists a natural holomorphic structure $\mathcal{E}_A$ on $E$ which induces a Dolbeault operator $\overline{\partial}_E$ satisfying that $A$ is the unique unitary connection such that $\overline{\partial}_A = \overline{\partial}_E$.*

Now, a complex bundle $E$ will admit several holomorphic structures, some of them isomorphic. These isomorphic holomorphic structures will give rise to *different* unitary connections. Isomorphism on holomorphic structures corresponds to the following equivalence on unitary connections:

*Definition* 2.76. Denote by $\mathscr{G}^c$ the group of all smooth complex automorphisms of $E$ covering the identity, called the *complex gauge group of $E$.*

The group $\mathscr{G}^c$ acts on $\overline{\partial}$-operators by conjugation, which induces an action on $\mathscr{A}^{1,1}$ as follows:

let $A \in \mathscr{A}^{1,1}$ and let $\bar{\partial}_A$ be the associated $\bar{\partial}$-operator (cf. Proposition 2.75). Then $g(A) \in \mathscr{A}^{1,1}$ is defined to be the Chern connection with respect to the $\bar{\partial}$-operator $g\bar{\partial}_A g^{-1} = \bar{\partial}_A - (\bar{\partial}_A g)g^{-1}$.

**Proposition 2.77** (Section 6.1.1 in [DK90]). *For $A \in \mathscr{A}^{1,1}$ denote by $\mathcal{E}_A$ the holomorphic structure given by Proposition 2.75. Then, the map*

$$\begin{aligned} \mathscr{A}^{1,1} &\longrightarrow \{\textit{holomorphic structures on } E\} \\ A &\mapsto \mathcal{E}_A \end{aligned}$$

*descends to a bijective map $\mathscr{A}^{1,1}/\mathscr{G}^c \longrightarrow \{\textit{holomorphic structures on } E\}/\simeq$, where $\mathcal{E} \simeq \mathcal{E}'$ if there exists a holomorphic map $f : \mathcal{E} \longrightarrow \mathcal{E}'$ covering the identity such that $f$ is an isomorphism of complex vector spaces in every fibre.*

In this sense, studying holomorphic structures on a vector bundle is essentially the same as fixing a hermitian metric and then studying unitary connections on that bundle. Later on, we will be interested in unitary connections with the following special curvature property:

*Definition* 2.78 (Hermite-Einstein connection). Let $X$ be a Kähler manifold of complex dimension $n$ with Kähler form $\omega \in \Omega^2(X)$. Let $E$ be a Hermitian vector bundle and $A$ be a unitary connection on $E$. Then $A$ is called a *Hermite-Einstein connection* (or *Hermitian-Yang-Mills connection*) if it satisfies the system of equations

$$F_A^{0,2} = 0 \text{ and } F_A \cdot \omega = \lambda \,\mathrm{Id} \tag{2.79}$$

for some constant $\lambda \in \mathbb{C}$. Here, $F_A \cdot \omega \in \Gamma(\mathrm{End}(E))$ is defined via $F_A \wedge \omega^{n-1} = (F_A \cdot \omega)\omega^n$. In particular, if $n = 2$, $2 \cdot F_A \cdot \omega = \langle F_A, \omega \rangle$.

All Chern connections satisfy the first of these conditions, i.e. $F_A^{0,2} = 0$, but they may not satisfy the condition $F_A \cdot \omega = \lambda \,\mathrm{Id}$. The following definition and theorem give a criterion for when a holomorphic bundle over a Kähler manifold of complex dimension two admits a hermitian metric so that its Chern connection is a Hermite-Einstein connection.

*Definition* 2.80 (Chern class of a coherent sheaf, [EH16]). Let $\mathscr{F}$ be a coherent sheaf over an

$n$-dimensional projective variety $X$ and let

$$0 \to \mathscr{E}_k \to \mathscr{E}_{k-1} \to \cdots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$$

be a locally free resolution. Then the total Chern class of $\mathscr{F}$ is defined as

$$c(\mathscr{F}) = \prod_{i=0}^{k} c(\mathscr{E}_i)^{(-1)^i} \in \Omega^*(X).$$

For this definition to make sense we need that all coherent sheaves $\mathscr{F}$ admit a locally free resolution, and that $c(\mathscr{F})$ does not depend of the choice of resolution. Both is proved in [Ful98, Section B.8].

*Definition* 2.81. Let $\mathscr{F}$ be a coherent sheaf over an $n$-dimensional projective variety $X$ with Kähler form $\omega$. Then, the *slope* of $\mathscr{F}$ is defined to be

$$\mu(\mathscr{F}) := \frac{\int_X c_1(\mathscr{F}) \wedge \omega^{n-1}}{\operatorname{rank}(\mathscr{F})}.$$

*Definition* 2.82 (Stable bundle). Let $\mathcal{E}$ be a holomorphic vector bundle over a projective variety $X$. Then $\mathcal{E}$ is called *stable*, if for any coherent subsheaf $\mathscr{F} \subset \mathcal{O}(E)$ with $0 < \operatorname{rank} \mathscr{F} < \operatorname{rank} \mathcal{E}$ the inequality

$$\mu(\mathscr{F}) < \mu(\mathcal{E})$$

holds.

**Theorem 2.83** (Theorem 1 in [Don85]). *A stable holomorphic vector bundle over a compact two-dimensional Kähler manifold admits a unique Hermitian metric so that its Chern connection is a Hermite-Einstein connection.*

As an example, consider the tangent bundle $E = T\mathbb{CP}^2$ of $\mathbb{CP}^2$. The complex projective space $\mathbb{CP}^2$ is a Kähler manifold, so it has a complex structure $J$. As for any other complex manifold, we have an isomorphism of complex vector bundles

$$\begin{aligned}\xi : E &\longrightarrow T^{1,0}\mathbb{CP}^2\\ v &\mapsto \frac{1}{2}(v - iJ(v)).\end{aligned}$$

$T^{1,0}\mathbb{CP}^2$ is a holomorphic vector bundle, and $\xi$ endows $E$ with a holomorphic structure via pullback. We denote $E$ together with this holomorphic structure by $\mathcal{E}$. We then have:

**Lemma 2.84** (Lemma 9.1.8 in [DK90]). *$\mathcal{E}$ is stable.*

Thus, from Theorem 2.83 we know that $\mathcal{E}$ admits some Hermitian metric so that its Chern connection is a Hermitian-Yang-Mills connection. We can exactly identify this Hermitian metric, too:

**Proposition 2.85.** *The Chern connection of the hermitian form induced by the Fubini-Study metric $g_{FS}$ on $\mathbb{CP}^n$ is a Hermite-Einstein connection.*

*Also, the Levi-Civita connection of the Fubini-Study metric is a Hermite-Einstein connection.*

*Proof.* Denote the Chern connection by $\nabla$. Then $F_A^{0,2} = 0$, just because it is a Chern connection. It remains to check the second part of Eq. (2.79). One checks through direct computation that $g_{\mathrm{FS}}$ is an Einstein metric satisfying

$$\mathrm{Ric} = (2n+2)g_{\mathrm{FS}} \tag{2.86}$$

(see [Pet16, Section 4.5.3]). The space $\mathbb{CP}^n$ is Kähler, and on any Kähler manifold we have that

$$\mathrm{Ric} = i \cdot \langle F_\nabla, \omega \rangle \tag{2.87}$$

viewed as endomorphisms of the tangent bundle (see [Huy05, Proposition 4.A.11]). The metric induces the identity endomorphism on the tangent bundle, so Eqs. (2.86) and (2.87) imply $F_\nabla \cdot \omega = \lambda\,\mathrm{Id}$ with $\lambda = -i(2n+2)$.

On a Kähler manifold, Levi-Civita connection and Chern connection agree, which proves the claim for the Levi-Civita connection. □

### 2.5.2 Rank 2 Vector Bundles

To every Hermitian vector bundle of rank 2 we can associate an SO(3)-bundle, which is explained in Proposition 2.90. We then revisit the tangent bundle on $\mathbb{CP}^2$ considered in the previous section and study its associated SO(3)-bundle.

*Definition* 2.88. The group $\mathrm{PU}(n) := \mathrm{U}(n)/C(\mathrm{U}(n))$ is called projective unitary group.

**Lemma 2.89**. *There is an isomorphism* $\mathrm{PU}(2) \simeq \mathrm{SO}(3)$ *of Lie groups.*

*Proof.* The group U(2) acts through the adjoint action on the space of trace-free Hermitian endomorphisms $\mathfrak{u}_0(2) \subset \mathfrak{u}(2)$. This action is isometric with respect to the metric given by the negative of the Killing form of $\mathfrak{u}(2)$ restricted to $\mathfrak{u}_0(2)$. Thus, PU(2) is a three-dimensional connected Lie group acting effectively and isometrically on a three-dimensional vector space, and thereby isomorphic to SO(3). □

**Proposition 2.90**. *Let $E$ be a complex vector bundle of rank 2 with hermitian metric $h$ over $X$. Denote its unitary frame bundle by* $\mathrm{U}(E)$. *Denote by* $\lambda : \mathrm{U}(2) \to \mathrm{PU}(2) \simeq \mathrm{SO}(3)$ *the quotient map and define*

$$P = \mathrm{U}(E) \times_\lambda \mathrm{SO}(3).$$

*Then, the characteristic classes of* $\mathfrak{u}_0(E)$ *and $E$ are related via*

$$p_1(\mathfrak{u}_0(E)) = c_1(E)^2 - 4c_2(E), \quad w_2(\mathfrak{u}_0(E)) = c_1(E) \textit{ mod } 2. \tag{2.91}$$

*Every connection $\nabla$ on $E$ canonically induces a connection on $P$. Furthermore, the connection on $P$ is an ASD instanton if $\nabla$ is a Hermite-Einstein connection.*

*Proof.* The bundle $P$ is defined as a principal bundle extension, and any connection can be canonically extended to any principal bundle extension. Assume that $\nabla$ is a Hermite-Einstein connection on $E$ and denote the induced connection on $P$ by $\widetilde{\nabla}$. We have that $[i \cdot \mathrm{Id}] = [0]$ in the quotient space $\mathrm{Lie}(\mathrm{PU}(2)) = \mathfrak{u}(2)/\mathrm{Lie}(C(\mathrm{U}(2)))$, therefore $\langle F_{\widetilde{\nabla}}, \omega\rangle = 0 \in \Omega^0(X, \mathrm{Ad}\, P)$. The $(0,2)$ and $(2,0)$ parts of the curvature satisfy $F^{0,2}_\nabla = F^{2,0}_\nabla = 0$, thus $F^{0,2}_{\widetilde{\nabla}} = F^{2,0}_{\widetilde{\nabla}} = 0$. The complexified space of self-dual 2-forms splits as $(\Omega^2_+)_{\mathbb{C}} = \Omega^{2,0} \oplus \langle \omega \rangle \oplus \Omega^{0,2}$, so $\widetilde{\nabla}$ is anti-self-dual. Equation (2.91) is [DK90, Eqn. 2.1.39]. ◻

As in Section 2.5.1, let $E = T\mathbb{CP}^2$.

**Proposition 2.92.** *Denote the* $\mathrm{SO}(3)$*-bundle associated to* $E$ *by means of Proposition 2.90 by* $F$ *and denote by* $\sigma : \mathbb{CP}^2 \to \mathbb{CP}^2$ *the complex conjugation on* $\mathbb{CP}^2$*. Then* $E$ *and* $\sigma^* E$ *are not isomorphic, while* $F$ *and* $\sigma^* F$ *are isomorphic.*

The proof uses:

**Theorem 2.93** (Theorem 14.10 in [MS74]). *The total Chern class of* $T\mathbb{CP}^n$ *is* $(1+a)^{n+1}$*, where* $a$ *is a suitably chosen generator of* $H^2(\mathbb{CP}^n, \mathbb{Z})$*.*

*Proof of Proposition 2.92.* We get from Theorem 2.93 and Eq. (2.91):

$$c_1(E) = 3a, c_2(E) = 3a^2, p_1(F) = -3a^2, w_2(F) = a \bmod 2,$$

where $a$ is a suitably chosen generator of $H^2(\mathbb{CP}^2, \mathbb{Z})$. Complex projective 2-space $\mathbb{CP}^2$ can be given the structure of a CW-complex with a single 2-cell

$$\mathbb{CP}^1 \simeq \{[x_0 : x_1 : 0] \in \mathbb{CP}^2\} \subset \mathbb{CP}^2$$

and no 1-cells and no 3-cells. Thus, $H^2(\mathbb{CP}^2, \mathbb{R})$ is generated by this $\mathbb{CP}^1$. The complex conjugation $\sigma$ restricts to $\mathbb{CP}^1$ and reverses its orientation, so acts as $-1$ on $H^2(\mathbb{CP}^2, \mathbb{Z})$, in particular $\sigma^* a = -a$. Therefore, $c_1(\sigma^* E) \neq c_1(E)$, which implies that $\sigma^* E$ and $E$ are not isomorphic. On the other hand, $p_1(\sigma^* F) = p_1(F)$ and $w_2(\sigma^* F) = w_2(F)$. So, by Theorem 2.35, we have that $F$ and $\sigma^* F$ are isomorphic. ◻

*Remark* 2.94. We will construct an explicit bundle isomorphism of $F$ and $\sigma^* F$ in Proposition 4.140. Thus, we will obtain Proposition 2.92 without the use of Theorem 2.35.

## 2.6 Gauge Theory on $G_2$-manifolds

*Definition* 2.95. Let $(Y, \varphi)$ be a $G_2$-manifold, $\psi = *_\varphi \varphi$, and $E$ be a principal bundle over $Y$. A connection $A \in \mathscr{A}(E)$ is called a *$G_2$-instanton*, if $F_A \in \Gamma(\Lambda^2_{14} \otimes \operatorname{Ad} E)$, i.e. (by Theorem 2.20)

$$F_A \wedge \psi = 0, \tag{2.96}$$

where the wedge product is taken in the 2-form part of $\Lambda^2 \otimes \operatorname{Ad} E$.

*Example* 2.97. Flat connections are $G_2$-instantons.

*Example* 2.98. Let $A$ be an ASD instanton on a bundle $E$ over a Hyperkähler 4-fold $X$. Denote by $p_X : \mathbb{R}^3 \times X \to X$ the projection onto the second factor. Then $\mathbb{R}^3 \times X$ carries the torsion-free $G_2$-structure $\varphi$ from Eq. (2.27), and $p_X^* A$ is a $G_2$-instanton on the bundle $p_X^* E$ with respect to this $G_2$-structure. To see this, let $\omega_1, \omega_2, \omega_3 \in \Omega^2(X)$ denote a Hyperkähler triple on $X$. These 2-forms are self-dual, thus $A$ being ASD is equivalent to $F_A \wedge \omega_i = 0$ for $i \in \{1, 2, 3\}$. Recall that for the product $G_2$-structure, we have that

$$*\varphi = \psi = \frac{1}{2}\omega_1^2 - \mathrm{d}x_{12} \wedge \omega_3 - \mathrm{d}x_{23} \wedge \omega_1 - \mathrm{d}x_{31} \wedge \omega_2$$

and therefore

$$F_{p_X^* A} \wedge \psi = p_X^*(F_A) \wedge \psi = 0.$$

A $G_2$-instanton $A$ satisfies $*(F_A \wedge \varphi) = -F_A$ by Theorem 2.20. Thus, if $\varphi$ is closed,

$$\mathrm{d}_A^* F_A = - * \mathrm{d}_A (F_A \wedge \varphi) = - * (\mathrm{d}_A F_A) \wedge \varphi$$

which vanishes due to the Bianchi identity. This means that $A$ is a critical point of the Yang-Mills energy functional

$$\begin{aligned} \mathrm{YM} : \mathscr{A}(E) &\to \mathbb{R} \\ A &\mapsto \int_Y |F_A|^2 \operatorname{vol}_Y . \end{aligned}$$

But even more is true:

**Proposition 2.99** (Proposition 1.97 in [Wal13a]). *Let $\varphi$ be a closed $G_2$-structure on $Y$. Then $G_2$-instantons with respect to $\varphi$ are absolute minima of the Yang-Mills functional.*

Later on, we will study the linearisation of the instanton equation. The linearisation at a point $A \in \mathscr{A}(E)$ of Eq. (2.96) is

$$\begin{aligned} l : \Omega^1(Y, \operatorname{Ad} E) &\to \Omega^1(Y, \operatorname{Ad} E) \\ a &\mapsto *(\psi \wedge \mathrm{d}_A a). \end{aligned} \tag{2.100}$$

This is not Fredholm (if the structure group $G$ is at least one-dimensional), because elements $u \in \mathscr{G}(E)$ of the gauge group satisfy $F_{u^*A} = u^* F_A$ and therefore preserve the $G_2$-instanton equation. Therefore, the infinitesimal action of the gauge group is in the kernel of $l$. As elliptic operators are Fredholm, that also implies $l$ is not an elliptic operator.

As we have seen in Section 2.4 it is customary to add in the *Coulomb gauge* condition $\mathrm{d}_A^* a = 0$ in order to make the linearised instanton operator elliptic. But in our case, $(l, \mathrm{d}_A^*) : \Omega^1(M, \operatorname{Ad} E) \to (\Omega^1 \oplus \Omega^0)(Y, \operatorname{Ad} E)$ cannot be elliptic either, because it is a map between vector bundles of different rank. This problem is overcome in the following proposition:

**Lemma 2.101** (Proposition 1.98 in [Wal13b]). *Let $(Y, \varphi)$ be a compact $G_2$-manifold, $\psi = *_\varphi \varphi$, and $E$ be a principal bundle over $Y$, and $A \in \mathscr{A}(E)$. Then $A$ is a $G_2$-instanton if and only if there exists $\xi \in \Omega^0(Y, \operatorname{Ad} E)$ such that*

$$*(F_A \wedge \psi) + \mathrm{d}_A \xi = 0. \tag{2.102}$$

So, for a fixed connection $A \in \mathscr{A}(E)$, $\xi \in \Omega^0(Y, \operatorname{Ad} E)$, and $a \in \Omega^1(Y, \operatorname{Ad} E)$ we consider the system

$$\begin{aligned} *(F_{A+a} \wedge \psi) + \mathrm{d}_{A+a} \xi &= 0 \\ \mathrm{d}_A^* a &= 0. \end{aligned} \tag{2.103}$$

Here, every solution $(\xi, a)$ defines the $G_2$-instanton $A + a$ which is in Coulomb gauge with respect to $A$. The linearisation of Eq. (2.103) is an elliptic operator:

**Proposition 2.104.** *The linearisation of Eq.* (2.103) *is*

$$
\begin{aligned}
L_A : (\Omega^0 \oplus \Omega^1)(Y, \operatorname{Ad} E) &\to (\Omega^0 \oplus \Omega^1)(Y, \operatorname{Ad} E) \\
\begin{pmatrix} \xi \\ a \end{pmatrix} &\mapsto \begin{pmatrix} 0 & \mathrm{d}_A^* \\ \mathrm{d}_A & *(\psi \wedge \mathrm{d}_A) \end{pmatrix} \begin{pmatrix} \xi \\ a \end{pmatrix}
\end{aligned}
\tag{2.105}
$$

*which is a self-adjoint elliptic operator if* $\mathrm{d}^* \varphi = 0$.

*Proof.* Denote $l = *(\psi \wedge \mathrm{d}_A) : \Omega^1(Y, \operatorname{Ad} E) \to \Omega^1(Y, \operatorname{Ad} E)$ and denote its dual by $l^*$. For $a, b \in \Omega^1(Y, \operatorname{Ad} E)$ we then have

$$
\langle a, l^* b \rangle \operatorname{vol} = \psi \wedge \mathrm{d}_A a \wedge b = \langle a, *d_A(\psi \wedge b) \rangle \operatorname{vol} = \langle a, *(\psi \wedge d_A b) \rangle \operatorname{vol}
$$

where we used $\mathrm{d}^* \varphi = 0$ in the last step. Thus, $l$ is self-adjoint which implies that $L_A$ is self-adjoint.

The operator $L_A$ is associated to the complex

$$
\Omega^0(Y, \operatorname{Ad} E) \xrightarrow{\mathrm{d}_A} \Omega^1(Y, \operatorname{Ad} E) \xrightarrow{l} \Omega^1(Y, \operatorname{Ad} E) \xrightarrow{\mathrm{d}_A^*} \Omega^0(Y, \operatorname{Ad} E). \tag{2.106}
$$

For $x \in Y$ and $0 \neq \xi \in T_x Y \simeq \mathbb{R}^7 \simeq (\mathbb{R}^7)^*$, the symbol of Eq. (2.106) applied to $\xi$ is then the sequence

$$
0 \to \Lambda^0 \otimes \mathfrak{g} \xrightarrow{(\cdot) \wedge \xi} \Lambda^1 \otimes \mathfrak{g} \xrightarrow{*(\psi \wedge (\cdot) \wedge \xi)} \Lambda^1 \otimes \mathfrak{g} \xrightarrow{\xi \lrcorner (\cdot)} \Lambda^0 \otimes \mathfrak{g} \to 0. \tag{2.107}
$$

It remains to check that this sequence is exact. The 4-form $\psi$ and the Hodge star are preserved by $G_2$ and $G_2$ acts transitively on $S^6 \subset \mathbb{R}^7$, so it suffices to check that Eq. (2.107) is exact for any (non-zero) choice of $\xi$, say $\xi = (1, 0, 0, 0, 0, 0, 0)$. This is then an explicit calculation that can be carried out using Eq. (2.18). □

*Remark* 2.108. A coordinate-free proof for the ellipticity of the complex in Eq. (2.106) is given in [RC98, Section 3, Lemma 4].

# 3 Resolutions of $G_2$-orbifolds

We now turn to the construction of resolutions of $G_2$-orbifolds, where we glue together the orbifold $G_2$-structure and the product $G_2$-structure on $\mathbb{R}^3 \times X_{\mathrm{EH}}$, where $X_{\mathrm{EH}}$ denotes the Eguchi-Hanson space as before. In particular, we will revisit the construction of [Joy96b]. Starting with the torus $T^7$, we write down an finite group $\Gamma$ that acts on $T^7$ and preserves the flat $G_2$-structure thereon. Following this, we construct smooth 7-manifolds $N_t$ carrying a 1-parameter family of $G_2$-structures $\varphi^t$, which are close to the flat $G_2$-structure, in a suitable sense. We then give a new proof for the fact that $\varphi^t$ can be perturbed to a torsion-free $G_2$-structure, and give an estimate for the size of the perturbation. This is stated in the main result of this section, Theorem 3.84:

**Theorem.** *Choose $\alpha \in (0,1)$ and $\beta \in (-1,0)$ both close to 0. Let $N_t$ be the resolution of $T^7/\Gamma$ from Eq.* (3.31) *and $\varphi^t \in \Omega^3(N_t)$ the $G_2$-structure with small torsion from Eq.* (3.33). *There exists $c > 0$ independent of $t$ such that the following is true: for $t$ small enough, there exists $\eta^t \in \Omega^2(N_t)$ such that $\widetilde{\varphi} = \varphi^t + \mathrm{d}\eta^t$ is a torsion-free $G_2$-structure, and $\eta^t$ satisfies*

$$\left\|\eta^t\right\|_{C^{2,\alpha/2}_{\beta;t}} \leq ct^{7/2-\beta}.$$

*In particular,*

$$\left\|\widetilde{\varphi} - \varphi^t\right\|_{L^\infty} \leq ct^{5/2} \text{ and } \left\|\widetilde{\varphi} - \varphi^t\right\|_{C^{0,\alpha/2}} \leq ct^{5/2-\alpha/2} \text{ as well as } \left\|\widetilde{\varphi} - \varphi^t\right\|_{C^{1,\alpha/2}} \leq ct^{3/2-\alpha/2}.$$

As is common in gluing constructions in differential geometry, we obtain this result by following the three step procedure of

1. Constructing an approximate solution (cf. Section 3.2.1)
2. Estimating the linearisation of the equation to be solved (cf. Section 3.2.3)
3. Perturbing the approximate solution to a genuine solution (cf. Section 3.2.4)

This method was first employed in [Tau82] for the construction of anti-self-dual connections over 4-manifolds. A similar but slightly simpler proof of the same results is given in [DK90, Section 7.2]. An expository article about this principle, which is in spirit close to the matter of this section, is [Don12].

## 3.1 Analysis on the Eguchi-Hanson Space

### 3.1.1 Harmonic forms on $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$

In this section, we will list homogeneous harmonic forms on $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$ with decay. Because $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$ is the cone over $\mathrm{SO}(3)$, we will see that such forms correspond to eigenforms on $\mathrm{SO}(3)$, and we will therefore review the spectral decomposition of the Laplacian on $S^3$ and $\mathrm{SO}(3)$.

We begin by defining cones and homogeneous forms on them.

*Definition* 3.1. For a Riemannian manifold $(\Sigma, g_\Sigma)$, the Riemannian manifold $C(\Sigma) = \Sigma \times \mathbb{R}_{>0}$ endowed with the metric $g_C = \mathrm{d}r^2 + r^2 g_\Sigma$ is called the *Cone over* $\Sigma$.

*Definition* 3.2. Let $\lambda \in \mathbb{R}$. Then $\gamma \in \Omega^k(C(\Sigma))$ is called *homogeneous of order* $\lambda$ if there exist $\alpha \in \Omega^{k-1}(\Sigma), \beta \in \Omega^k(\Sigma)$ such that

$$\gamma = r^{\lambda+k}\left(\frac{dr}{r} \wedge \alpha + \beta\right).$$

*Remark* 3.3. For $t \in \mathbb{R}_{>0}$ denote by $(\cdot t) : C(\Sigma) \to C(\Sigma)$ the dilation map given by $(\cdot t)(r, \sigma) = (tr, \sigma)$ for $(r, \sigma) \in C(\Sigma)$. Then, if $\gamma \in \Omega^k(C(\Sigma))$ is homogeneous of order $\lambda$, we have $(\cdot t)^* |\gamma|_{g_C} = t^\lambda |\gamma|_{g_C}$.

Homogeneous harmonic forms do not exist for all orders and we make the following definition:

*Definition* 3.4. For a cone $C = C(\Sigma)$, denote by $\Delta_{k,C}$ the Laplacian acting on $k$-forms on $C$. The set

$$\mathcal{D}_{\Delta_{k,C}} = \{\lambda \in \mathbb{R} : \exists \gamma \in \Omega^k(C), \gamma \neq 0, \text{ homogeneous of order } \lambda \text{ with } \Delta_{k,C}\gamma = 0\}$$

is called the set of *critical rates of* $\Delta_{k,C}$.

It will turn out that critical rates are intimately related to harmonic forms on Eguchi-Hanson space. This is the content of the next section and we will see the set $\mathcal{D}_{\Delta_{k,C}}$ appear again there. The purpose of this section is to describe $\mathcal{D}_{\Delta_{1,C(\mathrm{SO}(3))}}$ and $\mathcal{D}_{\Delta_{2,C(\mathrm{SO}(3))}}$, which is achieved in Proposition 3.10. We prepare the proposition by putting some results for harmonic forms on Riemannian cones in place:

**Lemma 3.5** (Lemma A.1 in [FHN20]). *Let $\gamma = r^{\lambda+k}\left(\frac{dr}{r} \wedge \alpha + \beta\right)$ be a $k$-form on $C(\Sigma)$ homogeneous of order $\lambda$. For every function $u = u(r)$ we have $\Delta(u\gamma) = r^{\lambda+k-2}\left(\frac{dr}{r} \wedge A + B\right)$, where*

$$A = u\Big(\triangle\alpha - (\lambda+k-2)(\lambda+n-k)\alpha - 2d^*\beta\Big) - r\dot{u}\,(2\lambda+n-1)\,\alpha - r^2\ddot{u}\,\alpha,$$
$$B = u\Big(\triangle\beta - (\lambda+n-k-2)(\lambda+k)\beta - 2d\alpha\Big) - r\dot{u}\,(2\lambda+n-1)\,\beta - r^2\ddot{u}\,\beta.$$

**Theorem 3.6** (Theorem A.2 in [FHN20]). *Let $\gamma = r^{\lambda+k}\left(\frac{dr}{r} \wedge \alpha + \beta\right)$ be a harmonic $k$-form on $C(\Sigma)$ homogeneous of order $\lambda$. Then $\gamma$ decomposes into the sum of homogeneous harmonic forms $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ where $\gamma_i = r^{\lambda+k}\left(\frac{dr}{r} \wedge \alpha_i + \beta_i\right)$ satisfies the following conditions.*

*(i)* $\beta_1 = 0$ *and* $\alpha_1$ *satisfies* $d\alpha_1 = 0$ *and* $\Delta\alpha_1 = (\lambda+k-2)(\lambda+n-k)\alpha_1$.

*(ii)* $(\alpha_2, \beta_2) \in \Omega^{k-1}_{coexact} \times \Omega^k_{exact}$ *satisfies the first-order system*

$$d\alpha_2 = (\lambda+k)\beta_2, \qquad d^*\beta_2 = (\lambda+n-k)\alpha_2.$$

*In particular, if $(\alpha_2, \beta_2) \neq 0$ then $\lambda+k \neq 0 \neq \lambda+n-k$ and the pair $(\alpha_2, \beta_2)$ is uniquely determined by either of the two factors, which is a coexact/exact eigenform of the Laplacian with eigenvalue $(\lambda+k)(\lambda+n-k)$.*

*(iii)* $(\alpha_3, \beta_3) \in \Omega^{k-1}_{coexact} \times \Omega^k_{exact}$ *satisfies the first-order system*

$$d\alpha_3 + (\lambda+n-k-2)\beta_3 = 0 = d^*\beta_3 + (\lambda+k-2)\alpha_3.$$

*In particular, if $(\alpha_3, \beta_3) \neq 0$ then $\lambda+k-2 \neq 0 \neq \lambda+n-k-2$ and the pair $(\alpha_3, \beta_3)$ is uniquely determined by either of the two factors, which is a coexact/exact eigenform of the Laplacian with eigenvalue $(\lambda+k-2)(\lambda+n-k-2)$.*

*(iv)* $\alpha_4 = 0$ *and* $\beta_4$ *satisfies* $d^*\beta_4 = 0$ *and* $\Delta\beta_4 = (\lambda+n-k-2)(\lambda+k)\beta_4$.

*The decomposition $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ is unique, except when $\lambda = -\frac{n-2}{2}$; in that case forms of type (ii) and (iii) coincide, and there is a unique decomposition $\gamma = \gamma_1 + \gamma_2 + \gamma_4$.*

The previous proposition relates harmonic forms on the cone $C(\mathrm{SO}(3))$ to eigenforms of the Laplacian on $\mathrm{SO}(3)$. The group $\mathrm{SO}(4)$ acts via pullback on complex-valued differential forms on $S^3$, and it turns out that the decomposition of this action into irreducible components gives the spectral decomposition for the Laplacian on $S^3$. This is made precise in the following two theorems, and as $S^3$ is a double cover of $\mathrm{SO}(3)$, we will get the spectral decomposition of the Laplacian on $\mathrm{SO}(3)$ from them.

**Theorem 3.7** (Theorem B in [Fol89]). *The complex-valued $L^2$-functions and 1-forms on $S^3$ decompose into the following irreducible $\mathrm{SO}(4)$-invariant subspaces:*

$$\Omega^0(S^3,\mathbb{C}) = \bigoplus_{m=1}^{\infty} \Phi_{0,m},$$
$$\Omega^1(S^3,\mathbb{C}) = \bigoplus_{m=1}^{\infty} \left(\Phi_{1,m} \oplus \Phi^-_{1,m} \oplus \Psi_{1,m}\right).$$

*Here, $\Phi_{0,m}, \Phi_{1,m}, \Phi^-_{1,m}, \Psi_{1,m}$ are defined as follows: denote by $j : S^3 \to \mathbb{R}^4$ the inclusion map and define $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$, and $\partial r = \sum_{j=1}^4 x_j \partial x_j$. Then let*

$$\Phi_{0,m} = j^*\mathscr{G}_{0,m+1}, \text{ where } \mathscr{G}_{0,m} \text{ is the smallest } \mathrm{SO}(4)\text{-inv. space containing } z_1^{m-1},$$
$$\Phi_{1,m} = j^*\mathscr{F}_{1,m}, \text{ where } \mathscr{F}_{1,m} \text{ is the smallest } \mathrm{SO}(4)\text{-inv. space containing } z_1^{m-1}\partial r \lrcorner (\mathrm{d}z_1 \wedge \mathrm{d}z_2).$$
$$\Phi^-_{1,m} = j^*\mathscr{F}^-_{1,m}, \text{ where } \mathscr{F}^-_{1,m} \text{ is the smallest } \mathrm{SO}(4)\text{-inv. space containing } z_1^{m-1}\partial r \lrcorner (\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_2}).$$
$$\Psi_{1,m} = j^*\mathscr{G}_{1,m}, \text{ where } \mathscr{G}_{1,m} \text{ is the smallest } \mathrm{SO}(4)\text{-inv. space containing } z_1^{m-1}\,\mathrm{d}z_1.$$

**Theorem 3.8** (Theorem C in [Fol89]). *Let $\Phi_{0,m}, \Phi_{1,m}, \Phi^-_{1,m}, \Psi_{1,m}$ as in Theorem 3.8. Then $\Phi_{0,m}$, $\Phi_{1,m} \oplus \Phi^-_{1,m}$, and $\Psi_{1,m}$ are eigenspaces for the Laplacian with eigenvalues $m(m+2)$, $(m+1)^2$, and $m(m+2)$ respectively.*

**Corollary 3.9.** *Let $S^3$ be endowed with the round metric and $\mathrm{SO}(3) = S^3/\{\pm 1\}$ be endowed with the quotient metric.*

1. *Then, the spectrum of the Laplacian $\Delta_{0,\mathrm{SO}(3)}$ acting on real-valued $L^2$-functions on $\mathrm{SO}(3)$*

*is:*

$$\mathrm{Spec}(\Delta_{0,\mathrm{SO}(3)}) = \{k(k+2) : k \in \mathbb{Z}_{\geq 0}, k \text{ even}\} = \{0, 8, 24, \dots\}.$$

2. *The smallest eigenvalue of the Laplacian $\Delta_{1,\mathrm{SO}(3)}$ acting on real-valued 1-forms with coefficients in $L^2$ on $\mathrm{SO}(3)$ is 4 and has multiplicity 6.*

*Proof of Corollary 3.9.*

1. This follows from Theorems 3.7 and 3.8 and the fact that functions in the space $\Phi_{0,m}$ defined in Theorem 3.7 are invariant under the antipodal map $(-1) : S^3 \longrightarrow S^3$ if and only if $m$ is even.

2. By Theorem 3.8, the smallest eigenvalue of the Laplacian acting on complex-valued 1-forms on $S^3$ is 3. We see from the explicit description of the eigenspace that the eigenforms are not invariant under the antipodal map. Thus, the eigenvalue 3 does not occur on $\mathrm{SO}(3)$.

   The next smallest eigenvalue is 4. It is realised, and it remains to check the dimension of its eigenspace: for the complex vector spaces defined in Theorem 3.7 we have $\Phi_{1,1} \simeq (\Lambda^2_+)^{\mathbb{C}}$ and $\Phi^-_{1,1} \simeq (\Lambda^2_-)^{\mathbb{C}}$, the complexification of (anti-)self-dual constant forms on $\mathbb{R}^4$. Here is how to see that $\Phi_{1,1} \simeq (\Lambda^2_+)^{\mathbb{C}}$, the other isomorphism is analogous. We have

   $$\mathrm{d}z_1 \wedge \mathrm{d}z_2 = \mathrm{d}x_{13} - \mathrm{d}x_{24} + i\,\mathrm{d}x_{23} + i\,\mathrm{d}x_{14} =: \omega.$$

   The element $g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{SO}(4)$ sends this to $-\mathrm{d}x_{13} + \mathrm{d}x_{24} + i\,\mathrm{d}x_{23} + i\,\mathrm{d}x_{14}$, so the smallest $\mathrm{SO}(4)$-invariant space containing $\omega$ must also contain the self-dual form $\mathrm{d}x_{13} - \mathrm{d}x_{24} = \frac{1}{2}(\omega - g\omega)$. Because $\Lambda^2_+$ is irreducible, this $\mathrm{SO}(4)$-invariant space must contain all of $(\Lambda^2_+)^{\mathbb{C}}$. Contracting with the radial vector field $\partial r$ and restricting to $S^3$ are $\mathrm{SO}(4)$-equivariant operations, one checks that the result is non-zero, and therefore

$\Phi_{1,1} \simeq \left(\Lambda^2_+\right)^{\mathbb{C}}$.

Altogether, $\Phi_{1,1}$ and $\Phi^-_{1,1}$ are representations of SO(4) of complex dimension 3. They consist of 1-forms on $S^3$ that are invariant under the antipodal map, which proves the claim.

□

We can now combine the results about harmonic forms on $C(\mathrm{SO}(3))$ with the spectral decomposition of the Laplacian on SO(3) to find the critical rates for the Laplacian on $C(\mathrm{SO}(3))$:

**Proposition 3.10.**

1. *There are no harmonic* 1*-forms on* $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$ *which are homogeneous of order* $\lambda$ *for* $\lambda \in [-2, 0)$. *In other words* $\mathcal{D}_{\Delta_{1,(\mathbb{C}^2\setminus\{0\})/\{\pm 1\}}} \cap [-2, 0) = \emptyset$.
2. *There is a six-dimensional space of harmonic* 2*-forms on* $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$ *which are homogeneous of order* $-2$.

   *There are no harmonic* 2*-forms on* $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$ *which are homogeneous of order* $\lambda$ *for* $\lambda \in (-2, 0)$.

*Proof.* It follows from point two in Proposition 2.5 that $C(\mathrm{SO}(3))$ and $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$ are isometric as Riemannian manifolds and we prove the statements on $C(\mathrm{SO}(3))$.

1. Let $\lambda \in [-2, 0)$ and assume there exists a harmonic homogeneous 1-form of order $\lambda$ on $C(\mathrm{SO}(3))$. We show that the 1-form must vanish by showing that forms satisfying any of the cases (i), (ii), (iii), and (iv) from Theorem 3.6 are zero. Using the notation from the theorem, we get the following:

   (i) In this case, $\Delta\alpha_1 = (\lambda - 1)(\lambda + 3)\alpha_1$. For $\lambda \in [-2, 0)$, the factor $(\lambda - 1)(\lambda + 3)$ is negative, so our assumption implies that $\alpha_1$ is a closed 0-form that is an eigenform of $\Delta_{\mathrm{SO}(3)}$ for a negative eigenvalue, which implies $\alpha_1 = 0$ by Corollary 3.9.

   (ii) In this case, $\beta_2$ is an exact 1-form with $\Delta_{\mathrm{SO}(3)}\beta_2 = (\lambda + 1)(\lambda + 3)\beta_2$. We have $(\lambda + 1)(\lambda + 3) < 8$ for $\lambda \in [-2, 0)$, and therefore $\beta_2 = 0$ as in case (i).

(iii) In this case, $\beta_3$ is an exact 1-form with $\Delta_{SO(3)}\beta_3 = (\lambda+1)(\lambda-3)\beta_3$, and $\beta_3 = 0$ follows as before.

(iv) In this case, $\beta_4$ is a co-closed 1-form with $\Delta_{SO(3)}\beta_3 = (\lambda+1)^2\beta_3$. For $\lambda \in [-2, 0)$, we have $(\lambda+1)^2 < 3$, and because of Corollary 3.9 this implies $\beta_4 = 0$.

2. Let $\lambda \in [-2, 0)$. Going through the cases (i), (ii), (iii), and (iv) from Theorem 3.6, we will find that there are six linearly independent harmonic homogeneous 2-forms of order $-2$ in case (iii), but no other harmonic homogeneous forms. Using the notation from the theorem, we get the following:

   (i) In this case, we get a 1-form that is an eigenform of the Laplacian on $SO(3)$ for the eigenvalue $\lambda(\lambda+2) < 0$, which must be 0 by Corollary 3.9.

   (ii) In this case, we get a 1-form that is an eigenform of the Laplacian on $SO(3)$ for the eigenvalue $(\lambda+2)^2 < 4$, which must be 0 by Corollary 3.9.

   (iii) In this case, we get a 1-form that is an eigenform of the Laplacian on $SO(3)$ for the eigenvalue $\lambda^2$. There are six of these by Corollary 3.9 for $\lambda = -2$ and none for $\lambda \in (-2, 0)$. In the case of $\lambda = -2$ all six eigenforms give rise to harmonic 2-forms of order $\lambda = -2$ on $C(SO(3))$.

   (iv) In this case, we get a 2-form $\beta_4$ that is an eigenform of the Laplacian on $SO(3)$ for the eigenvalue $(\lambda+2)^2 < 4$. The Hodge dual $*\beta_4$ is then a 1-form that is an eigenform for the same eigenvalue, which must be 0 by Corollary 3.9.

□

For an application later we will not only need to know how many harmonic homogeneous forms there are, but also how many harmonic homogeneous forms *with* $\log(r)$ *coefficients* there are. Often, these two notions coincide, and the following proposition asserts that this is also the case in our setting.

*Definition* 3.11. Let $\Sigma$ be a connected Riemannian manifold and $C = C(\Sigma)$ its cone. For $\lambda \in \mathbb{R}$, define

$$\mathcal{K}(\lambda)_{\Delta_{p,C(\Sigma)}} = \left\{ \begin{array}{l} \gamma = \sum_{j=0}^{m} (\log r)^j \gamma_j \text{ for } m \geq 0,\, \gamma_j \in \Omega^p(C(\Sigma)), \text{ such that} \\ \Delta_{p,C(\Sigma)} \gamma = 0, \text{ where each } \gamma_j \text{ is homogeneous of order } \lambda \end{array} \right\}.$$

**Proposition 3.12.** *Let* $\gamma = \sum_{j=0}^{m} (\log r)^j \gamma_j \in \mathcal{K}(-2)_{\Delta_{2,C(\Sigma)}}$, *then* $\gamma_j = 0$ *for* $j > 0$.

*Proof.* Write $\gamma_j = r^{\lambda+k}\left(\frac{dr}{r} \wedge \alpha_j + \beta_j\right)$. Then, by Lemma 3.5, for $j \geq 1$,

$$\Delta(\log(r)^j \gamma_j) = r^{-2}\left(\frac{\mathrm{d}r}{r} \wedge A + B\right), \text{ where}$$

$$A = \underbrace{\log(r)^j (\Delta\alpha_j - 2\,\mathrm{d}^*\beta_j)}_{=0} + 2j\log(r)^{j-1}\alpha_j - j(j-1)\log(r)^{j-2}\alpha_j, \tag{3.13}$$

$$B = \underbrace{\log(r)^j (\Delta\beta_j - 2\,\mathrm{d}\alpha_j)}_{=0} + 2j\log(r)^{j-1}\beta_j - j(j-1)\log(r)^{j-2}\beta_j. \tag{3.14}$$

Here, the terms $\Delta\alpha_j - 2\,\mathrm{d}^*\beta_j$ and $\Delta\beta_j - 2\,\mathrm{d}\alpha_j$ vanish, because $\alpha_j$ is coexact and satisfies $2\beta_j = \mathrm{d}\alpha_j$, and $\beta_j$ is exact and satisfies $\mathrm{d}^*\beta_j = 2\alpha_j$ according to the discussion of point 2 of Proposition 3.10. The term $\Delta\gamma$ is a polynomial in $\log(r)$, and the condition $\Delta\gamma = 0$ prescribes that all coefficients of that polynomial vanish. Assume that $m > 0$ and check the coefficient of $\log(r)^{m-1}$: Eq. (3.13) implies that $\alpha_m = 0$ and Eq. (3.14) implies that $\beta_m = 0$, i.e. $\gamma_m = 0$. Repeating the argument, we find that $\gamma_{m-1} = 0, \gamma_{m-2} = 0, \ldots, \gamma_2 = 0, \gamma_1 = 0$, which is what we wanted to show. ☐

### 3.1.2 Harmonic forms on Eguchi-Hanson Space

In the previous section we looked at certain harmonic forms on $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$. The Eguchi-Hanson space $X_{\mathrm{EH}}$ is asymptotic to the cone $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$, and we can say a great deal about harmonic forms on $X_{\mathrm{EH}}$ just from knowing the harmonic forms on $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$. This is a consequence of the work of Lockhart and McOwen (cf. [LM85, Loc87]) and will be the content of this section.

We will want statements about harmonic forms in certain weighted Hölder spaces. These spaces are defined in the following:

*Definition* 3.15. Define the weight functions

$$\begin{aligned} w : X_{\mathrm{EH}} &\to \mathbb{R}_{\geq 0} & \qquad w : X_{\mathrm{EH}} \times X_{\mathrm{EH}} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto 1 + |\rho(x)|, & (x, y) &\mapsto \min\{w(x), w(y)\}. \end{aligned}$$

Let $U \subset X_{\mathrm{EH}}$. For $\alpha \in (0,1)$, $\beta \in \mathbb{R}$, $k \in \mathbb{N}$, and $f \in \Omega^k(X_{\mathrm{EH}})$ define the *weighted Hölder norm of $f$* via

$$\begin{aligned} [f]_{C^{0,\alpha}_{\beta}(U)} &:= \sup_{\substack{x,y \in U \\ d_{g_{(1)}}(x,y) \leq w(x,y)}} w(x,y)^{\alpha-\beta} \frac{|f(x) - f(y)|_{g_{(1)}}}{d_{g_{(1)}}(x,y)^{\alpha}}, \\ ||f||_{L^{\infty}_{\beta}(U)} &:= \left\| w_t^{-\beta} f \right\|_{L^{\infty}(U)}, \\ ||f||_{C^{k,\alpha}_{\beta}(U)} &:= \sum_{j=0}^{k} \left\| \nabla^j f \right\|_{L^{\infty}_{\beta-j}(U)} + \left[ \nabla^j f \right]_{C^{0,\alpha}_{\beta-j}(U)} \end{aligned}$$

The term $f(x) - f(y)$ in the first line denotes the difference between $f(x)$ and the parallel transport of $f(y)$ to the fibre $\Omega^k(X_{\mathrm{EH}})|_x$ along one of the shortest geodesics connecting $x$ and $y$. When $U$ is not specified, take $U = X_{\mathrm{EH}}$.

Throughout the article we will set $\beta$ to be a negative number. Informally, an element in the $C^{k,\alpha}_{\beta}$ Hölder space decays like $d_{g_{(1)}}(\cdot, \rho^{-1}(0))^{\beta}$, as $d_{g_{(1)}}(\cdot, \rho^{-1}(0)) \to \infty$.

We will now make the meaning of $X_{EH}$ *being asymptotic to a cone* precise.

*Definition* 3.16. Let $\Sigma$ be a connected Riemannian manifold and $C = C(\Sigma)$ be its cone with cone metric $g_C$. A Riemannian manifold $(M, g_M)$ is called *asymptotically conical with cone $C$ and rate $\nu < 0$* if there exists a compact subset $L \subset M$, a number $R > 0$, and a diffeomorphism $\phi : (R, \infty) \times \Sigma \to M \setminus L$ satisfying

$$|\nabla^k(\phi^*(g_M) - g_C)|_{g_C} = O(\varrho^{\nu-k}) \text{ for all } k \geq 0 \text{ as } \varrho \to \infty.$$

Here, $\nabla$ denotes the Levi-Civita connection with respect to $g_C$ and $\varrho : (0,\infty) \times \Sigma \to (0, \infty)$ is the projection onto the first component.

**Proposition 3.17.** *The Eguchi-Hanson space $X_{EH}$ endowed with the metric $g_{(1)}$ is asymptotically* *conical with cone $C = C(\mathrm{SO}(3))$ and rate $\nu = -4$.*

*Proof.* This is the second point of Proposition 2.10. □

We then have the following results about harmonic forms in $L^2$ on Eguchi-Hanson space:

**Lemma 3.18.**

1. *We have $H^2_{sing}(X_{EH}) = H^2_{deRham}(X_{EH}) = \mathbb{R}$. For $k > 0$ define $\nu_k \in \Omega^2(X_{EH})$ to be*

$$\nu_k := f_k(r)^{-6} r\, \mathrm{d}r \wedge \eta^1 - f_k(r)^{-2} \eta^2 \wedge \eta^3 \tag{3.19}$$

   *and endow $X_{EH}$ with the metric $g_{(k)}$. Then $\nu_k \in L^2(\Lambda^2(X_{EH}))$, $\Delta_{g_{(k)}} \nu_k = 0$, $[\nu_k]$ generates $H^2_{deRham}(X_{EH})$, and $\nu_k$ is the unique element in $L^2(\Lambda^2(X_{EH})) \cap [\nu_k]$ satisfying $\Delta_{g_{(k)}} \nu_k = 0$. Moreover, $\nu_1 \in C^{2,\alpha}_{-4}(\Lambda^2(X_{EH}))$. Away from the exceptional orbit $\rho^{-1}(0) \simeq S^2$, we have that*

$$\nu_k = \mathrm{d}\lambda_k, \text{ where } \lambda_k = -f_k(r)^{-2} \eta^1.$$

2. *The $L^2$-kernels of $\Delta_{g_{(k)}}$ acting on forms of different degrees are as follows:*

$$\mathrm{Ker}(\Delta_{g_{(k)}} : L^2(\Lambda^2(X_{EH})) \to L^2(\Lambda^2(X_{EH}))) = \langle \nu_k \rangle,$$
$$\mathrm{Ker}(\Delta_{g_{(k)}} : L^2(\Lambda^p(X_{EH})) \to L^2(\Lambda^p(X_{EH}))) = 0 \text{ for } p \neq 2.$$

   *For $k = 1$ and $\beta \in [-4, -2)$ they coincide with the $C^{2,\alpha}_{\beta}$-kernels.*

*Proof.*

1. We have that $X_{\mathrm{EH}} = T^*S^2$ as smooth manifolds, therefore $H^2_{\mathrm{sing}}(X_{\mathrm{EH}}) = \mathbb{R}$. On smooth manifolds $H^2_{\mathrm{sing}}(X_{\mathrm{EH}}) = H^2_{\mathrm{deRham}}(X_{\mathrm{EH}})$ by de Rham's Theorem.

   One checks with a direct computation that $\nu_k$ from Eq. (3.19) is closed and anti-self-dual, and therefore co-closed. The equality $\nu_k = \mathrm{d}\lambda_k$ follows from a direct computation as well.

For $k = 0$, Eq. (3.19) still defines an element $\nu_0 \in \Omega^2(\mathbb{C}^2/\{\pm 1\} \setminus \{0\})$. One checks through direct calculation that $\nu_0 \in C^{2,\alpha}_{-4}(\Lambda^2(\mathbb{C}^2/\{\pm 1\}))$. Using the fact that $X_{\mathrm{EH}}$ is asymptotically locally Euclidean (cf. Proposition 2.10), one gets the Hölder estimate on $X_{\mathrm{EH}}$. Furthermore, $C^{2,\alpha}_{-4} \subset L^\infty_{-4} \subset L^2$, so $\nu_k$ is an element in $L^2(\Lambda^2(\mathbb{C}^2/\{\pm 1\}))$.

By Poincaré duality, we have $H^2_{\mathrm{cs}}(X_{\mathrm{EH}}) = H^2_{\mathrm{sing}}(X_{\mathrm{EH}}) = \mathbb{R}$, where $H^2_{\mathrm{cs}}(X_{\mathrm{EH}})$ denotes the de Rham cohomology with compact support. [Loc87, Example (0.15)] and [Loc87, Theorem (7.9)] give that the map

$$\begin{aligned}\mathcal{H}^2(X_{\mathrm{EH}}) := \{\xi \in L^2(\Lambda^2 T^* X_{\mathrm{EH}}) : \mathrm{d}\xi = \mathrm{d}^*\xi = 0\} &\to \mathrm{Im}\left(H^2_{\mathrm{cs}}(X_{\mathrm{EH}}) \hookrightarrow H^2_{\mathrm{deRham}}(X_{\mathrm{EH}})\right)\\ \xi &\mapsto [\xi]\end{aligned}$$

is an isomorphism. Thus $[\nu_k]$ generates $H^2_{\mathrm{deRham}}(X_{\mathrm{EH}})$ and $\nu_k \in [\nu_k]$ is the unique element in $L^2(\Lambda^2(X_{\mathrm{EH}})) \cap [\nu_k]$ satisfying $\mathrm{d}\nu_k = 0$, $\mathrm{d}^*\nu_k = 0$.

It remains to check that $\nu_k$ is also the unique element in $L^2(\Lambda^2(X_{\mathrm{EH}})) \cap [\nu_k]$ satisfying $\Delta_{g(k)}\nu_k = 0$. The equations $\Delta_{g(k)}\nu_k = 0$ and $(\mathrm{d}+\mathrm{d}^*)\nu_k = 0$ are equivalent by the same integration by parts argument as in the compact case, namely for $M > 0$:

$$\begin{aligned}&\int_{\{r\le M\}} \langle (\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d})\nu_k, \nu_k\rangle \,\mathrm{dvol}_{g(k)}\\
&= \int_{\{r\le M\}} \langle (\mathrm{d}\mathrm{d}^*)\nu_k, \nu_k\rangle \,\mathrm{dvol}_{g(k)} + \int_{\{r\le M\}} \langle (\mathrm{d}^*\,\mathrm{d})\nu_k, \nu_k\rangle \,\mathrm{dvol}_{g(k)}\\
&= \int_{\{r\le M\}} \langle \mathrm{d}^*\nu_k, \mathrm{d}^*\nu_k\rangle \,\mathrm{dvol}_{g(k)} + \int_{\{r\le M\}} \mathrm{d}(\mathrm{d}^*\nu_k \wedge *\nu_k)\\
&\quad + \int_{\{r\le M\}} \langle \mathrm{d}\nu_k, \mathrm{d}\nu_k\rangle \,\mathrm{dvol}_{g(k)} + \int_{\{r\le M\}} \mathrm{d}(\nu_k \wedge * \,\mathrm{d}\nu_k)\\
&= \int_{\{r\le M\}} \left(\langle \mathrm{d}^*\nu_k, \mathrm{d}^*\nu_k\rangle + \langle \mathrm{d}\nu_k, \mathrm{d}\nu_k\rangle\right) \mathrm{dvol}_{g(k)}\\
&\quad + \int_{\partial\{r\le M\}} \left(\mathrm{d}^*\nu_k \wedge *\nu_k + \nu_k \wedge * \,\mathrm{d}\nu_k\right),\end{aligned}$$

where we used $\mathrm{d}(\mathrm{d}^*\nu_k \wedge *\nu_k) = \mathrm{d}\mathrm{d}^*\nu_k \wedge *\nu_k - \mathrm{d}^*\nu_k \wedge \mathrm{d}*\nu_k$ in the second step, and Stokes' Theorem in the last step. The last term tends to $0$ as $M \to \infty$, because of the decay of elements in $C^{2,\alpha}_{-4;t}(\Lambda^2(X_{\mathrm{EH}}))$. So, $\Delta_{g(k)}\nu_k = 0$ implies that $\mathrm{d}^*\nu_k = 0$, $\mathrm{d}\nu_k = 0$, and the converse implication is trivial.

2. The first line is a restatement of the previous point. The other lines are [Loc87, Example

(0.15)] with proof in [Loc87, Theorem (7.9)].

The $L^2$-kernels coincide with the $C^{2,\alpha}_\beta$-kernels, as $C^{2,\alpha}_\beta(\Lambda^p(X_{\mathrm{EH}}))$ embeds into $L^2(\Lambda^p(X_{\mathrm{EH}}))$ for $\beta < -2$ and the explicit description of the $L^2$-kernels shows that all kernel elements are actually in $C^{2,\alpha}_\beta(\Lambda^p(X_{\mathrm{EH}}))$ for $\beta \geq -4$.

□

*Remark* 3.20. Note that $v_k$ from the lemma cannot have compact support by the unique continuation property for elliptic equations. We only have that $[v_k]$ contains a form of compact support.

The previous lemma makes statements about the $L^2$-kernels of the Laplacian on $X_{\mathrm{EH}}$ acting on $p$-forms. Using the results from the previous section about harmonic forms on $\mathbb{C}^2/\{\pm 1\}$, we can rule out additional harmonic forms even in some of the weighted Hölder spaces that do not embed into $L^2$. The key proposition that will be proved throughout the rest of this section is the following:

**Proposition 3.21.** *For $\beta \in (-4, 0)$, the kernels of the $\Delta_{g_{(1)}}$ acting on forms in $C^{2,\alpha}_\beta$ of different degrees are as follows:*

$$\mathrm{Ker}(\Delta_{g_{(1)}} : C^{2,\alpha}_\beta(\Lambda^2(X_{EH})) \to C^{0,\alpha}_{\beta-2}(\Lambda^2(X_{EH}))) = \langle v_1 \rangle,$$
$$\mathrm{Ker}(\Delta_{g_{(1)}} : C^{2,\alpha}_\beta(\Lambda^p(X_{EH})) \to C^{0,\alpha}_{\beta-2}(\Lambda^p(X_{EH}))) = 0 \textit{ for } p \neq 2.$$

The connection between the Laplacian on Eguchi-Hanson space and its cone is described in the following results taken from [KL20, Section 4] which were developed in [LM85, Loc87]. The theory works for a much bigger class of operators, but we will only reproduce it for the Laplacian here.

*Definition* 3.22. Let $M$ be asymptotically conical and let the notation be as in Definition 3.16. Denote by $\varrho : C(\Sigma) \to \mathbb{R}_{\geq 0}$ the radius function, and use the same symbol to denote a map from $M$ to $\mathbb{R}_{>0}$ that agrees with $\phi_* \varrho$ on $\phi(R, \infty) \subset M$. Let $E$ be a vector bundle with metric and metric connection $\nabla$ over $M$. Then, for $1 > p > \infty$, $l \geq 0$, $\lambda \in \mathbb{R}$ denote by $L^p_{l,\lambda}$ the completion of $C^\infty_{\mathrm{cs}}(E)$ with respect to the norm

$$||\gamma||_{L^p_{l,\lambda}} = \left(\sum_{j=0}^{l}\int_M |\varrho^{-\lambda+j}\nabla^j\gamma|^p\varrho^{-4}\,\mathrm{vol}_M\right)^{1/p}.$$

The space $L^p_{l,\lambda}$ is called the *$L^p$-Sobolev space with $l$ derivatives and decay faster than $\lambda$.*

**Theorem 3.23** (Theorem 4.10 in [KL20]). *For $\lambda \in \mathbb{R}$, denote by $\Delta_{p,g_{(1)}} : L^q_{2,\lambda}(\Lambda^p(X_{EH})) \to L^q_{0,\lambda-2}(\Lambda^p(X_{EH}))$ the Laplacian of the metric $g_{(1)}$ acting on $p$-forms. Then,* $\operatorname{Ker}\Delta_{p,g_{(1)}}$ *is invariant under changes of $\lambda$, as long as we do not hit any critical rates. That is, if the interval $[\lambda,\lambda']$ is contained in the complement of $\mathcal{D}_{\Delta_{p,(\mathbb{C}^2\setminus\{0\})/\{\pm1\}}}$, then*

$$\begin{aligned}&\operatorname{Ker}\left(\Delta_{p,g_{(1)}} : L^q_{2,\lambda}(\Lambda^p(X_{EH})) \to L^q_{0,\lambda-2}(\Lambda^p(X_{EH}))\right)\\ =&\operatorname{Ker}\left(\Delta_{p,g_{(1)}} : L^q_{2,\lambda'}(\Lambda^p(X_{EH})) \to L^q_{0,\lambda'-2}(\Lambda^p(X_{EH}))\right).\end{aligned}$$

**Proposition 3.24** (Theorem 4.20 in [KL20]). *Let $\lambda_1 < \lambda_2$ such that $\mathcal{K}(\lambda_i)_{\Delta_{p,C(\Sigma)}} = 0$ for $i \in \{1,2\}$. Then, the maps*

$$\begin{aligned}&\Delta_{p,g_{(1)},L^2_{l+2,\lambda_1}} : L^2_{l+2,\lambda_1}(\Lambda^p(X_{EH})) \to L^2_{l,\lambda_1-2}(\Lambda^p(X_{EH}))\\ \textit{and } &\Delta_{p,g_{(1)},L^2_{l+2,\lambda_2}} : L^2_{l+2,\lambda_2}(\Lambda^p(X_{EH})) \to L^2_{l,\lambda_2-2}(\Lambda^p(X_{EH}))\end{aligned}$$

*are Fredholm and the difference in their indices is given by*

$$\operatorname{ind}\left(\Delta_{p,g_{(1)},L^2_{l+2,\lambda_2}}\right) - \operatorname{ind}\left(\Delta_{p,g_{(1)},L^2_{l+2,\lambda_1}}\right) = \sum_{\lambda\in\mathcal{D}_{\Delta_{(\mathbb{C}^2\setminus\{0\})/\{\pm1\}}}\cap(\lambda_1,\lambda_2)} \dim\mathcal{K}(\lambda)_{\Delta_{p,(\mathbb{C}^2\setminus\{0\})/\{\pm1\}}} \tag{3.25}$$

Combining everything, we get the following characterisation of harmonic forms with decay:

**Theorem 3.26**. *For $\lambda \in (-4,0)$, the $L^2_{2,\lambda}$-kernels of $\Delta_{p,g_{(1)}}$ acting on $p$-forms of different degrees are the same as the $L^2$-kernels, namely:*

$$\begin{aligned}\operatorname{Ker}(\Delta_{g_{(1)}} : L^2_{2,\lambda}(\Lambda^2(X_{EH})) \to L^2_{0,\lambda-2}(\Lambda^2(X_{EH}))) &= \langle\nu_1\rangle,\\ \operatorname{Ker}(\Delta_{g_{(1)}} : L^2_{2,\lambda}(\Lambda^p(X_{EH})) \to L^2_{0,\lambda-2}(\Lambda^p(X_{EH}))) &= 0 \textit{ for } p \neq 2.\end{aligned}$$

*Proof.* 0-forms and 4-forms: it follows from the maximum principle that every harmonic function that decays at infinity must vanish. The Hodge star is an isomorphism between 0-forms and 4-forms that commutes with the Laplacian, so the statement for 0-forms implies that statement for 4-forms.

1-forms and 3-forms: the kernel of the Laplacian is zero for rate $-2$ by the second point of Lemma 3.18. By the first point of Proposition 3.10, there are no critical rates in the interval $[-2, 0)$. So, Theorem 3.23 implies the claim for 1-forms. As above, we get the statement for 3-forms by using the Hodge star.

2-forms: by Proposition 3.10 the only critical rate in $[-2, 0)$ is $-2$. The kernel of the Laplacian on 2-forms stays the same for rates $\lambda \in (-4, -2)$ by Lemma 3.18. By Theorem 3.23, the dimension of the kernel of the Laplacian acting on 2-forms with decay $\lambda \in (-4, 0)$ may therefore only change at $\lambda = -2$. We know from Propositions 3.12 and 3.24 that the index increases by six when crossing the critical rate $\lambda = -2$. We will now check that the dimension of the cokernel decreases by 6, which implies that the dimension of the kernel does not change.

The dual space of $L^2_{0,\lambda}$ is $L^2_{0,-4-\lambda}$. Therefore, the cokernel of $\Delta_{g_{(1)}} : L^2_{2,-2}(\Lambda^2(X_{\mathrm{EH}})) \to L^2_{0,-4}(\Lambda^2(X_{\mathrm{EH}}))$ is isomorphic to the kernel of the adjoint operator $\Delta^*_{g_{(1)}} = \Delta_{g_{(1)}} : L^2_{2,0}(\Lambda^2(X_{\mathrm{EH}})) \to L^2_{0,-2}(\Lambda^2(X_{\mathrm{EH}}))$. Here we used that elements in the cokernel of $\Delta_{g_{(k)}}$ are smooth by elliptic regularity, so it does not matter how many derivatives we demand for sections acted on by the adjoint operator.

We now explicitly write down six linearly independent harmonic forms in $L^2_{2,0}(\Lambda^2(X_{\mathrm{EH}}))$: three of them are the (self-dual) Kähler forms $\omega_1^{(1)}$, $\omega_2^{(1)}$, and $\omega_3^{(1)}$ defined in Proposition 2.5.

Analogously, we can define three harmonic *anti-self-dual* forms with respect to $g_{(k)}$ for all $k > 0$. To this end, extend $\eta^1, \eta^2, \eta^3 \in \mathfrak{so}(3)$ from Proposition 2.5 to *right*-invariant forms on $\mathrm{SO}(3)$, denoted by $\hat{\eta}_1$, $\hat{\eta}_2$, $\hat{\eta}_3$. These forms satisfy $d\hat{\eta}_1 = -\hat{\eta}^2 \wedge \hat{\eta}^3$ etc. In analogy to Proposition 2.5, define

$$\hat{e}^1(r) = r f_k^{-1}(r)\hat{\eta}^1, \qquad \hat{e}^2(r) = f_k(r)\hat{\eta}^2, \qquad \hat{e}^3(r) = f_k(r)\hat{\eta}^3$$

and

$$\hat{\omega}_1^{(k)} = \mathrm{d}t \wedge \hat{e}^1 - \hat{e}^2 \wedge \hat{e}^3, \qquad \hat{\omega}_2^{(k)} = \mathrm{d}t \wedge \hat{e}^2 - \hat{e}^3 \wedge \hat{e}^1, \qquad \hat{\omega}_3^{(k)} = \mathrm{d}t \wedge \hat{e}^3 - \hat{e}^1 \wedge \hat{e}^2.$$

One checks through computation that $\hat{\omega}_i^{(k)}$ are closed and anti-self-dual, and therefore harmonic. A priori, they are defined on $\mathbb{R}_{>0} \times \mathrm{SO}(3)$, and it remains to check that they extend to all of $X_{\mathrm{EH}}$. We have $\hat{\omega}_2^{(k)} = \mathrm{d}(r\hat{\eta}^2)$ and $\hat{\omega}_3^{(k)} = \mathrm{d}(r\hat{\eta}^3)$, where $r\hat{\eta}^2$ and $r\hat{\eta}^3$ are well-defined 1-forms on all of $X_{\mathrm{EH}}$, because they vanish as $r \to 0$. Therefore, $\hat{\omega}_2^{(k)}$ and $\hat{\omega}_3^{(k)}$ are well-defined on $X_{\mathrm{EH}}$.

We have that $\hat{\omega}_1^{(k)} = rf_k^{-2}(r)\,\mathrm{d}r \wedge \hat{\eta}^1 - f_k^{-2}(r)\hat{\eta}^2 \wedge \hat{\eta}^3$, where the first summand vanishes as $r \to 0$, and the second summand is a multiple of the volume form on $\mathrm{SO}(3) \times_{\mathrm{SO}(2)} \{0\} \simeq S^2$ pulled back under the projection

$$\begin{aligned} \mathrm{SO}(3) \times_{\mathrm{SO}(2)} V &\to \mathrm{SO}(3) \times_{\mathrm{SO}(2)} V \\ (g, x) &\mapsto (g, 0). \end{aligned}$$

Thus $\hat{\omega}_1^{(k)}$ is also defined on all of $X_{\mathrm{EH}}$. The forms $\eta^1, \eta^2, \eta^3, \hat{\eta}^1, \hat{\eta}^2, \hat{\eta}^3$ are linearly independent which implies that $\omega_1^{(k)}, \omega_2^{(k)}, \omega_3^{(k)}, \hat{\omega}_1^{(k)}, \hat{\omega}_2^{(k)}, \hat{\omega}_3^{(k)}$ are linearly independent.

Last, note that for each $g \in \mathrm{SO}(3)$ we can express $\hat{\eta}^i(g)$ as a linear combination of $\eta^i(g)$. Each $\eta^i$ decays like $r^{1/2}$ as $r \to \infty$, which shows that the $\hat{\omega}_i^{(k)}$ have the same decay as the Hyperkähler triple $\omega_i^{(k)}$, which is covariant constant. Thus, we have that $\omega_i^{(1)}, \hat{\omega}_i^{(1)} \in L^2_{2,0}(\Lambda^2(X_{\mathrm{EH}}))$, but $\notin L^2_{2,-\epsilon}(\Lambda^2(X_{\mathrm{EH}}))$ for all $\epsilon > 0$ and $i \in \{1, 2, 3\}$.

Therefore, the dimension of the cokernel of $\Delta_{g_{(1)}} : L^2_{2,\lambda}(\Lambda^2(X_{\mathrm{EH}})) \to L^2_{0,\lambda-2}(\Lambda^2(X_{\mathrm{EH}}))$ changes by six when crossing the critical rate $\lambda = -2$, and the dimension of the kernel stays the same. □

Proposition 3.21 is now an immediate consequence of Theorem 3.26.

*Proof of Proposition 3.21.* For $\epsilon > 0$ we have that $C^{2,\alpha}_{\beta-\epsilon}$ is embedded in $L^2_{2,\beta}$, so the claim follows from Theorem 3.26. □

## 3.2 Torsion-Free $G_2$-Structures on the Generalised Kummer Construction

In the two articles [Joy96b], Joyce constructed the first examples of manifolds with holonomy equal to $G_2$. One starts with the flat 7-torus, which admits a flat $G_2$-structure. A quotient of the torus by maps preserving the $G_2$-structure still carries a flat $G_2$-structure, but has *singularities*. The maps are carefully chosen, so that the singularities are modelled on $T^3 \times \mathbb{C}^2/\{\pm 1\}$. By the results of Section 3.1, $T^3 \times \mathbb{C}^2/\{\pm 1\}$ has a family of resolutions $T^3 \times X_{\text{EH}} \to T^3 \times \mathbb{C}^2/\{\pm 1\}$ of one real parameter, where $X_{\text{EH}}$ denotes the Eguchi-Hanson space, and the parameter defines the size of a minimal sphere in $X_{\text{EH}}$. We can define a smooth manifold by gluing these resolutions over the singularities in the quotient of the torus.

The product manifold $T^3 \times X_{\text{EH}}$ carries the product $G_2$-structure from Eq. (2.27). That means we have two torsion-free $G_2$-structures on our glued manifold: one coming from flat $T^7$, and the product $G_2$-structure near the resolution of the singularities. We will interpolate between the two to get one globally defined $G_2$-structure. This will no longer be torsion-free, but it will have small enough torsion in the sense of Theorem 2.26. This is the argument that was used in [Joy96b] to prove the existence of a torsion-free $G_2$-structure, and the construction of this $G_2$-structure with small torsion is the content of Section 3.2.1.

Sections 3.2.2 to 3.2.4 give an alternative proof of the existence of a torsion-free $G_2$-structure on this glued manifold.

### 3.2.1 Resolutions of $T^7/\Gamma$

We briefly review the generalised Kummer construction as explained in [Joy96b]. Let $(x_1, \dots, x_7)$ be coordinates on $T^7 = \mathbb{R}^7/\mathbb{Z}^7$, where $x_i \in \mathbb{R}/\mathbb{Z}$, endowed with the flat $G_2$-structure $\varphi_0$ from Definition 2.17. Let $\alpha, \beta, \gamma : T^7 \to T^7$ defined by

$$\begin{aligned}
\alpha &: (x_1, \dots, x_7) \mapsto (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7), \\
\beta &: (x_1, \dots, x_7) \mapsto \left(-x_1, \frac{1}{2} - x_2, x_3, x_4, -x_5, -x_6, x_7\right), \\
\gamma &: (x_1, \dots, x_7) \mapsto \left(\frac{1}{2} - x_1, x_2, \frac{1}{2} - x_3, x_4, -x_5, x_6, -x_7\right).
\end{aligned} \tag{3.27}$$

Denote $\Gamma := \langle \alpha, \beta, \gamma \rangle$. The next lemmata collect some information about the orbifold $T^7/\Gamma$:

**Lemma 3.28** (Section 2.1 in part I, [Joy96b]). *$\alpha, \beta, \gamma$ preserve $\varphi_0$, we have $\alpha^2 = \beta^2 = \gamma^2 = 1$, and $\alpha, \beta, \gamma$ commute. We have that $\Gamma \simeq \mathbb{Z}_2^3$.*

**Lemma 3.29** (Lemma 2.1.1 in part I, [Joy96b]). *The elements $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, and $\alpha\beta\gamma$ of $\Gamma$ have no fixed points on $T^7$. The fixed points of $\alpha$ in $T^7$ are 16 copies of $T^3$, and the group $\langle \beta, \gamma \rangle$ acts freely on the set of 16 3-tori fixed by $\alpha$. Similarly, the fixed points of $\beta, \gamma$ in $T^7$ are each 16 copies of $T^3$, and the groups $\langle \alpha, \gamma \rangle$ and $\langle \alpha, \beta \rangle$ act freely on the sets of 16 3-tori fixed by $\beta, \gamma$ respectively.*

**Lemma 3.30** (Lemma 2.1.2 in part I, [Joy96b]). *The singular set $L$ of $T^7/\Gamma$ is a disjoint union of 12 copies of $T^3$. There is an open subset $U$ of $T^7/\Gamma$ containing $L$, such that each of the 12 connected components of $U$ is isometric to $T^3 \times \left(B^4_\zeta/\{\pm 1\}\right)$, where $B^4_\zeta$ is the open ball of radius $\zeta$ in $\mathbb{R}^4$ for some positive constant $\zeta$ ($\zeta = 1/9$ will do).*

We now define a compact 7-manifold $M$, which can be thought of as a resolution of the orbifold $T^7/\Gamma$, and a one-parameter family of closed $G_2$-structures $\varphi^t$ thereon. We can choose an identification $U \simeq L \times \left(B^4_\zeta/\{\pm 1\}\right)$ such that we can write on $U$

$$\varphi_0 = \delta_1 \wedge \delta_2 \wedge \delta_3 - \sum_{i=1}^{3} \omega_i \wedge \delta_i, \qquad *\varphi_0 = \frac{1}{2}\omega_1 \wedge \omega_1 - \sum_{\substack{(i,j,k)=(1,2,3) \\ \text{and cyclic permutation}}} \omega_i \wedge \delta_j \wedge \delta_k,$$

where $\delta_1, \delta_2, \delta_3$ are covariant constant orthonormal 1-forms on $L$, and $\omega_1, \omega_2, \omega_3$ are the Hyperkähler triple from Definition 2.4, cf. Section 2.3.2.

As before, denote by $X_{\mathrm{EH}}$ the Eguchi-Hanson space and by $\rho : X_{\mathrm{EH}} \to \mathbb{C}^2/\{\pm 1\}$ the blowup map from Remark 2.13. Define $\check{r} := |\rho| : X_{\mathrm{EH}} \to \mathbb{R}_{\geq 0}$. For $t \in (0,1)$, let $\hat{U} := \hat{U}_t := L \times \{x \in X_{\mathrm{EH}} : \check{r}(x) < \zeta t^{-1}\}$. Define

$$N_t := \left((T^7/\Gamma) \setminus L \sqcup \hat{U}\right)/\sim, \tag{3.31}$$

where for $x = (x_h, x_v) \in U \subset L \times \mathbb{C}^2/\{\pm 1\}$ and $y = (y_h, y_v) \in \hat{U} \subset L \times X_{\mathrm{EH}}$ we have $x \sim y$ if $x_h = y_h$ and $t \cdot \rho(y_v) = x_v$. The smooth manifold $N_t$ also comes with a natural projection map

$\pi : N_t \longrightarrow T^7/\Gamma$ induced by $\rho$, and we extend $\check{r}$ to a map on all of $N_t$ via

$$\begin{aligned}\check{r} : N_t &\longrightarrow \mathbb{R}_{\geq 0}\\ x &\mapsto \begin{cases} |\rho(x)| & \text{if } x \in \hat{U}, \\ t^{-1}\zeta & \text{otherwise.}\end{cases}\end{aligned}$$

Write $r_t := t\check{r}$ and choose a non-decreasing function $\chi : [0, \zeta] \longrightarrow [0, 1]$ such that $\chi(s) = 0$ for $s \leq \zeta/4$ and $\chi(s) = 1$ for $s \geq \zeta/2$, and set

$$\widetilde{\omega}_i := \omega_i^{(1)} - \mathrm{d}\Big(\chi(r_t)\tau_i^{(1)}\Big). \tag{3.32}$$

The $\tau_i^{(1)}$ were defined in Proposition 2.10, and are the difference between the flat Hyperkähler triple on $\mathbb{C}^2/\{\pm 1\}$ and the Hyperkähler triple $(\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)})$ on $X_{\mathrm{EH}}$. On $\hat{U}$ we have $\widetilde{\omega}_i = \omega_i$ where $r_t > \zeta/2$, and $\widetilde{\omega}_i = \omega_i^{(1)}$ where $r_t < \zeta/4$. Now define a 3-form $\varphi^t \in \Omega^3(M)$ and a 4-form $\vartheta^t \in \Omega^4(N_t)$ as follows: on $(T^7/\Gamma) \setminus U \subset N_t$, set $\varphi^t = \varphi$ and $\vartheta = *\varphi$. On $\hat{U} \subset L \times X_{\mathrm{EH}}$ let

$$\varphi^t := \delta_1 \wedge \delta_2 \wedge \delta_3 - t^2 \sum_{i=1}^{3} \widetilde{\omega}_i \wedge \delta_i, \tag{3.33}$$

$$\vartheta^t := t^4 \frac{1}{2} \widetilde{\omega}_1 \wedge \widetilde{\omega}_1 - t^2 \sum_{\substack{(i,j,k)=(1,2,3)\\ \text{and cyclic permutation}}} \widetilde{\omega}_i \wedge \delta_j \wedge \delta_k. \tag{3.34}$$

This definition mimics the product situation explained in Section 2.3.2. For small $t$, the 3-form $\varphi^t$ is a $G_2$-structure and therefore induces a metric $g^t$. Both $\varphi^t$ and $\vartheta^t$ are closed forms, so, if $*\varphi^t = \vartheta^t$, then $\varphi^t$ would be a torsion-free $G_2$-structure by Theorem 2.22. However, this does not hold, and $\varphi^t$ is not a torsion-free $G_2$-structure. The following 3-form $\psi^t$ is meant to measure the torsion of $\varphi^t$:

$$*\psi^t = \Theta(\varphi^t) - \vartheta^t. \tag{3.35}$$

Its crucial properties are:

**Lemma 3.36.** *Let $\psi^t \in \Omega^3(M)$ as in Eq. (3.35). There exists a positive constant $c$ independent of $t$*

*such that*

$$\mathrm{d}^*\psi^t = \mathrm{d}^*\varphi^t, \qquad\qquad \|\psi^t\|_{C^{1,\alpha}} \le ct^4,$$

*where the Hölder norm is defined with respect to the metric $g^t$ and its induced Levi-Civita connection.*

*Proof.* The equality $\mathrm{d}^*\psi^t = \mathrm{d}^*\varphi^t$ follows from Eq. (3.35) and the fact that $\vartheta^t$ is closed.

The operator $*$ is parallel, so the covariant derivative $\nabla_X$ and $*$ commute for every vector field $X$ on $N_t$, therefore it suffices to estimate $*\psi^t$ rather than $\psi^t$. Write $\varphi^{(t)}_{X_{\mathrm{EH}}\times L} := \delta_1 \wedge \delta_2 \wedge \delta_3 - t^2 \sum_{i=1}^3 \omega_i^{(1)} \wedge \delta_i$ for the product $G_2$-structure on $X_{\mathrm{EH}} \times L$ and denote the induced metric, which is the product metric, by $g^{(t)}_{X_{\mathrm{EH}}\times L}$. Recall the linear map $T$ and the non-linear map $F$ from Proposition 2.24 satisfying $\Theta(\varphi+\xi) = *\varphi - T(\xi) - F(\xi)$ for a $G_2$-structure $\varphi$ and a small deformation $\xi$. Using this notation, we get:

$$\begin{aligned}
\Theta(\varphi^t) - \vartheta^t &= \Theta\left(\varphi^{(t)}_{X_{\mathrm{EH}}\times L} - t^2\delta_1 \wedge \mathrm{d}\Big(\chi(r_t)\tau_1^{(1)}\Big)\right) \\
&\quad - *_{g^{(t)}_{X_{\mathrm{EH}}\times L}} \varphi^{(t)}_{X_{\mathrm{EH}}\times L} + t^2\delta_2 \wedge \delta_3 \wedge \mathrm{d}\Big(\chi(r_t)\tau_1^{(1)}\Big) \\
&= T\left(t^2\delta_1 \wedge \mathrm{d}\Big(\chi(r_t)\tau_1^{(1)}\Big)\right) - F\left(-t^2\delta_1 \wedge \mathrm{d}\Big(\chi(r_t)\tau_1^{(1)}\Big)\right) \\
&\quad + t^2\delta_2 \wedge \delta_3 \wedge \mathrm{d}\Big(\chi(r_t)\tau_1^{(1)}\Big).
\end{aligned}$$

Here we used the equality $\omega_1^{(k)} - \omega_1 = \mathrm{d}\tau_1^{(k)}$ from Proposition 2.10 in the first step and the definition of $T$ and $F$ in the second step.

Note that $\Theta(\varphi^t) - \vartheta^t$ is supported on $\{x \in M : (\zeta/4)t^{-1} < \check{r} < (\zeta/2)t^{-1}\}$. Therefore, by Eq. (2.11),

$$\begin{aligned}
\left|t^2\,\mathrm{d}\Big(\chi(r_t)\tau_1^{(1)}\Big)\right|_{t^2 g_{(1)}} &\le \left|t^2\left(\mathrm{d}\chi(r_t)\right)\tau_1^{(1)}\right|_{t^2 g_{(1)}} + \left|t^2\chi(r_t)\,\mathrm{d}\tau_1^{(1)}\right|_{t^2 g_{(1)}} \\
&\le ct\left|t\tau_1^{(1)}\right|_{t^2 g_{(1)}} + c\left|t^2\chi(r_t)\,\mathrm{d}\tau_1^{(1)}\right|_{t^2 g_{(1)}} \\
&= tO(\check{r}^{-3}) + O(\check{r}^{-4}) \le ct^4.
\end{aligned}$$

Using the estimates for $T$ and $F$ from Proposition 2.24 we get the claim. $\square$

### 3.2.2 The Laplacian on $\mathbb{R}^3 \times X_{\mathrm{EH}}$

In the next section we will prove an estimate for the Laplacian on 2-forms on $N_t$. We will use a blowup argument to essentially reduce the analysis on $N_t$ to the analysis on $T^7/\Gamma$ and $\mathbb{R}^3 \times X_{\mathrm{EH}}$. In this section we will cite a general result for uniformly elliptic operators on product manifolds $\mathbb{R}^n \times Y$ from [Wal13b], where $Y$ is a Riemannian manifold, and use this to find that harmonic 2-forms on $\mathbb{R}^3 \times X_{\mathrm{EH}}$ are wedge products of parallel forms on $\mathbb{R}^3$ and harmonic forms on $X_{\mathrm{EH}}$.

*Definition* 3.37 (Definition 2.75 in [Wal13b]). A Riemannian manifold $Y$ is said to be of *bounded geometry* if it is complete, its Riemann curvature tensor is bounded from above and its injectivity radius is bounded from below. A vector bundle over $Y$ is said to be of *bounded geometry* if it has trivialisations over balls of fixed radius such that the transition functions and all of their derivatives are uniformly bounded. We say that a complete oriented Riemannian manifold $X$ has *subexponential volume growth* if for each $x \in X$ the function $r \mapsto \mathrm{vol}(B_r(x))$ grows subexponentially, i.e., $\mathrm{vol}(B_r(x)) = o(\exp(cr))$ as $r \to \infty$ for every $c > 0$.

**Lemma 3.38** (Lemma 2.76 in [Wal13b]). *Let $E$ be a vector bundle of bounded geometry over a Riemannian manifold $Y$ of bounded geometry and with subexponential volume growth, and suppose that $D : C^\infty(Y, E) \to C^\infty(Y, E)$ is a uniformly elliptic operator of second order whose coefficients and their first derivatives are uniformly bounded, that is non-negative, i.e., $\langle Da, a\rangle \geq 0$ for all $a \in W^{2,2}(Y, E)$, and formally self-adjoint. Let $p : \mathbb{R}^n \times Y \to Y$ be the projection onto the second component and $a \in C^\infty(\mathbb{R}^n \times Y, p^*E)$ such that*

$$(\Delta_{\mathbb{R}^n} + p^*D)\, a = 0$$

*and $||a||_{L^\infty}$ is finite, then $a$ is constant in the $\mathbb{R}^n$-direction, that is $a(x, y) = a(y)$. Here, $\Delta_{\mathbb{R}^n}$ acts on a section $a \in C^\infty(\mathbb{R}^n \times Y, p^*E)$ by using the identification $C^\infty(\mathbb{R}^n \times Y, p^*E) = C^\infty(\mathbb{R}^n, C^\infty(Y, E))$.*

**Corollary 3.39.** *Let $Y$ be a manifold of bounded geometry and with subexponential volume* *growth. If $a \in \Omega^2(\mathbb{R}^3 \times Y)$ satisfies $||a||_{L^\infty} < \infty$ and*

$$\Delta_{g_{\mathbb{R}^3} \oplus g_{(1)}} a = 0,$$

*then $a$ is a sum of terms of the form $a_1 \wedge a_2$, where $a_1 \in \Omega^k(\mathbb{R}^3)$ is parallel, and $a_2 \in \Omega^l(Y)$ satisfies $\Delta_{g_{(1)}} a_2 = 0$.*

*Proof.* We can view the vector bundle of 2-forms over $\mathbb{R}^3 \times Y$ as a pullback bundle pulled back from $Y$ via

$$\Lambda^2(\mathbb{R}^3 \times Y) \simeq p^* \left(\Lambda^2(Y) \oplus \Lambda^1(Y) \otimes \underline{\Lambda^1(\mathbb{R}^3)} \oplus \underline{\Lambda^2(\mathbb{R}^3)}\right)$$

where $\underline{\Lambda^k(\mathbb{R}^3)}$ denotes the trivial vector bundle over $Y$ whose fibre at each point is $\Lambda^k(\mathbb{R}^3)$. Under this identification, $\Delta_{\mathbb{R}^3 \times Y} = \Delta_{\mathbb{R}^3} + p^*(\Delta_Y + \Delta)$, where $\Delta$ is the canonical Laplacian on trivial vector bundles.

So, if $a \in \Omega^2(\mathbb{R}^3 \times Y)$ with $||a||_{L^\infty} < \infty$ and $\Delta_{g_{\mathbb{R}^3} \oplus g_{(1)}} a = 0$, then $a$ is the pullback of a section of $\Lambda^2(Y) \oplus \Lambda^1(Y) \otimes \underline{\Lambda^1(\mathbb{R}^3)} \oplus \underline{\Lambda^2(\mathbb{R}^3)}$ over $Y$ which is in the kernel of $\Delta_Y + \Delta$ by Lemma 3.38. Elements in the kernel of $\Delta_Y + \Delta$ over $Y$ are of the form $a_1 \wedge a_2$, where $a_1 \in \Omega^k(\mathbb{R}^3)$ is harmonic, and $a_2 \in \Omega^l(Y)$ satisfies $\Delta_{g_{(1)}} a_2 = 0$. Bounded harmonic $k$-forms on $\mathbb{R}^3$ can be identified with tuples of harmonic functions on $\mathbb{R}^3$ which are constant by the maximum principle. This means that the bounded harmonic $k$-forms are parallel which proves the claim. □

### 3.2.3 The Laplacian on $N_t$

We now move on to the heart of the argument: an operator bound for the inverse of the Laplacian on $N_t$. The Laplacian on 2-forms has a kernel of dimension $b^2(N_t)$, so we can only expect such a bound for forms which are not in the kernel. Standard elliptic theory would give an estimate for forms orthogonal to the kernel. This estimate would depend on the gluing parameter $t$, but we want a *uniform* estimate, i.e. an estimate independent of $t$. Proving such an estimate is the content of this section.

**Stating the estimate** We first define weighted Hölder norms analogous to the previous sections. These norms have the following two important properties: far away from $L$, they are uniformly equivalent to ordinary Hölder norms, and near $L$ they are uniformly equivalent to the weighted Hölder norms on $\mathbb{R}^3 \times X_{\mathrm{EH}}$, after applying a rescaling map.

*Definition* 3.40. For $t \in (0,1)$ define the weight functions

$$\begin{aligned} w_t : N_t &\to \mathbb{R}_{>0} \\ x &\mapsto t + r_t, \end{aligned} \tag{3.41}$$

$$\begin{aligned} w_{\mathbb{R}^3\times\mathbb{R}^4} : \mathbb{R}^3 \times \mathbb{R}^4 &\to \mathbb{R}_{>0} \\ (x,y) &\mapsto |y|, \\ w_{\mathbb{R}^3\times X_{\mathrm{EH}}} : \mathbb{R}^3 \times X_{\mathrm{EH}} &\to \mathbb{R}_{>0} \\ x &\mapsto 1 + \check{r} \end{aligned}$$

and for $k \in \mathbb{N}$, $\alpha \in (0,1)$, $\beta \in \mathbb{R}$ the weighted Hölder norms $||\cdot||_{C^{k,\alpha}_{\beta;t}}$ on $N_t$ and $||\cdot||_{C^{k,\alpha}_{\beta}}$ on $\mathbb{R}^3 \times \mathbb{R}^4$ and $\mathbb{R}^3 \times X_{\mathrm{EH}}$ respectively as in Definition 3.15.

We now define a way to decompose elements $a \in \Omega^2(N_t)$ into a component $\overline{\pi}_t$ that looks like $\nu_1 \in \Omega^2(X_{\mathrm{EH}})$ from Eq. (3.19) on every fibre $\{y\} \times X_{\mathrm{EH}} \subset T^3 \times X_{\mathrm{EH}}$, and a remainder, denoted by $\rho_t$. The reason for this is the following: the Laplacian on $\operatorname{Im}\overline{\pi}_t$ is approximately the Laplacian on $L$, and its inverse has operator norm of order $O(1)$ uniformly in $t$ as a map $C^{2,\alpha}_{\beta;t}(\Lambda^2(N_t)) \to C^{0,\alpha}_{\beta;t}(\Lambda^2(N_t))$. Notice that the weight does not change when applying the Laplacian. On $\operatorname{Im}\rho_t$, it will turn out that the Laplacian has operator norm of order $O(1)$ uniformly in $t$ as a map $C^{2,\alpha}_{\beta;t}(\Lambda^2(N_t)) \to C^{2,\alpha}_{\beta-2;t}(\Lambda^2(N_t))$. Here the weight changed in the same way as it did on the non-compact asymptotically conical space $X_{\mathrm{EH}}$, cf. Section 3.1.2. In order to prove an estimate of the form $||a|| \leq c\,||\Delta a||$ we will define norms that incorporate these two different scaling behaviours in this section. The idea is taken from [Wal17].

Let $\overline{\nu} \in \Omega^2(X_{\mathrm{EH}})$ be harmonic and with unit $L^2$-norm with respect to the norm $g_{(1)}$ on $X_{\mathrm{EH}}$. As a shorthand, write $\chi_t := \chi(2r_t)$. Define $\pi_t : \Omega^2(N_t) \to \Omega^0(L)$ via

$$(\pi_t a)(y) := \langle a|_{\{y\}\times X_{\mathrm{EH}}}, \chi_t \nu\rangle_{L^2, t^2 g_{X_{\mathrm{EH}}}} \text{ for } y \in L, \tag{3.42}$$

where $\nu \in \Omega^2(X_{\mathrm{EH}})$ is a multiple of $\overline{\nu}$ satisfying $\langle \chi_t \nu, \chi_t \nu \rangle_{L^2, t^2 g_{X_{\mathrm{EH}}}} = 1$. This is equivalent to $\langle \chi_t \nu, \chi_t \nu \rangle_{L^2, g_{X_{\mathrm{EH}}}} = 1$, i.e. in the metric $g_{X_{\mathrm{EH}}}$ rather than $t^2 g_{X_{\mathrm{EH}}}$, because the $L^2$-norm on 2-forms is a conformal invariant. Define $\iota_t : \Omega^0(L) \to \Omega^2(N_t)$ via

$$(\iota_t g) := \chi_t \cdot p_L^* g \cdot p_{X_{\mathrm{EH}}}^* \nu, \tag{3.43}$$

where $g \in \Omega^0(L)$, and $p_L : L \times X_{\mathrm{EH}} \to T^3$, $p_{X_{\mathrm{EH}}} : L \times X_{\mathrm{EH}} \to X_{\mathrm{EH}}$ are projection maps. As written, $(\iota_t g)$ is an element in $\Omega^2(L \times X_{\mathrm{EH}})$, but because $\mathrm{supp}(\iota_t g) \subset \hat{U}$, we can view it as an element in $\Omega^2(N_t)$. Then

$$\pi_t \iota_t g = g \text{ for all } g \in \Omega^0(L). \tag{3.44}$$

Last, define $\overline{\pi}_t := \iota_t \pi_t$, as well as $\rho_t := 1 - \overline{\pi}_t$.

**Proposition 3.45.** *For all $k \in \mathbb{N}$ and $\beta > -4$ there exists $c > 0$ independent of $t$ such that for all $g \in \Omega^0(L)$ we have that*

$$||\iota_t g||_{C^{k,\alpha}_{\beta;t}} \le c t^{-2-\beta} \, ||g||_{C^{k,\alpha}} . \tag{3.46}$$

*Proof.* For the $L^\infty$-norm we have that

$$\begin{aligned} \left\| p_L^* g \cdot p_{X_{\mathrm{EH}}}^* \nu \right\|_{L^\infty_{-4;t}, g_{N_t}} &\le c \left\| p_L^* g \cdot p_{X_{\mathrm{EH}}}^* \nu \cdot (t + t\check{r})^4 \right\|_{L^\infty, g_{\mathbb{R}^3} \oplus t^2 g_{X_{\mathrm{EH}}}} \\ &\le c \left\| p_L^* g \cdot p_{X_{\mathrm{EH}}}^* \nu \cdot (1 + \check{r})^4 t^4 t^{-2} \right\|_{L^\infty, g_{\mathbb{R}^3} \oplus g_{X_{\mathrm{EH}}}} \\ &\le c t^2 \left\| p_L^* g \right\|_{L^\infty} \end{aligned}$$

where we used that $\nu = O(\check{r}^{-4})$ and therefore

$$\left\| \nu \cdot \check{r}^4 \right\|_{L^\infty, g_{X_{\mathrm{EH}}}} \le c, \tag{3.47}$$

in the last step. For $\beta > -4$ we have that $||\chi_t||_{L^\infty_{4-\beta}} \le c t^{-4-\beta}$, which proves the claim for the weighted $L^\infty$-norm. The proof for higher derivatives is analogous. □

**Proposition 3.48.** *For all $k \in \mathbb{N}, \beta < 0$ there exists $c > 0$ independent of $t$ such that for all*

*$a \in \Omega^2(N_t)$ we have that*

$$||\pi_t a||_{C^{k,\alpha}} \leq t^{2+\beta-\alpha-k} ||a||_{C^{k,\alpha}_{\beta;t}} . \tag{3.49}$$

*Proof.* We first estimate the $L^\infty$-part, i.e. $||\pi_t a||_{L^\infty}$. To this end

$$\begin{aligned}
|\pi_t a(x)| &\leq \int_{\{x \in X_{\mathrm{EH}}: \check{r}(x) \leq t^{-1}\zeta\}} |a|_{t^2 g_{X_{\mathrm{EH}}}} \cdot |\nu|_{t^2 g_{X_{\mathrm{EH}}}} \mathrm{vol}_{t^2 g_{X_{\mathrm{EH}}}} \\
&\leq t^2 ||a||_{L^\infty_{\beta;t}} \int_{X_{\mathrm{EH}}} (t + \check{r}t)^\beta \cdot |\nu|_{g_{X_{\mathrm{EH}}}} \mathrm{vol}_{g_{X_{\mathrm{EH}}}} \\
&\leq ct^{2+\beta} ||a||_{L^\infty_{\beta;t}} \int_{X_{\mathrm{EH}}} (1+\check{r})^\beta \cdot (1+\check{r})^{-4} \mathrm{vol}_{g_{X_{\mathrm{EH}}}} \\
&\leq ct^{2+\beta} \underbrace{\int_0^\infty (1+\check{r})^{-4+\beta} \cdot \check{r}^3 \, \mathrm{d}\check{r}}_{\leq c} \\
&\leq ct^{2+\beta} ||a||_{L^\infty_{\beta;t}} ,
\end{aligned}$$

where in the second step we used the definition of $||\cdot||_{L^\infty_{\beta;t}}$ and switched from measuring in $t^2 g_{X_{\mathrm{EH}}}$ to measuring in $g_{X_{\mathrm{EH}}}$ which introduces the factor of $t^2$; in the third step we used $|\nu|_{g_{X_{\mathrm{EH}}}} \leq c(1+\check{r})^{-4}$; in the fourth step we used polar coordinates to switch from integrating over $X_{\mathrm{EH}}$ to integrating over $[0, \infty)$. The estimates for the Hölder norm, derivatives, and for other weights are proved analogously. □

We are now ready to define the composite norms which weigh the $\overline{\pi}_t$ and $\rho_t$ components differently.

*Definition* 3.50. For $\alpha \in (0,1)$ and $\beta \in (-1,0)$ let

$$\begin{aligned}
||a||_{\mathfrak{X}_t} &:= ||\rho_t a||_{C^{2,\alpha}_{\beta;t}} + t^{-3/2} ||\pi_t a||_{C^{2,\alpha}} , \\
||a||_{\mathfrak{Y}_t} &:= ||\rho_t a||_{C^{0,\alpha}_{\beta-2;t}} + t^{-3/2} ||\pi_t a||_{C^{0,\alpha}} .
\end{aligned}$$

In the following, we will always assume that $\alpha$ and $\beta$ are close to 0. The most restrictive estimate in which this fact is used is Eq. (3.81). For concreteness, one may choose $\alpha = 1/16$ and $\beta = -1/16$.

*Definition* 3.51 (Approximate kernel). Let $C_1, \ldots, C_{12}$ be the connected components of $\hat{U}$ and let $\chi_{C_i}$ be the characteristic function of the set $C_i$. Then define the *approximate kernel of* $\Delta$ *on* $N_t$ to be

$$\mathcal{K} := \{(1-\chi_t)\pi^* a : a \in \operatorname{Ker} \Delta_{T^7/\Gamma}\} \oplus \operatorname{span}\left(\chi_t \cdot p^*_{X_{\mathrm{EH}}} \nu \cdot \chi_{C_i}\right)_{i=1,\ldots,12},$$

where $\pi : N_t \longrightarrow T^7/\Gamma$ is the projection map from the previous section.

**Proposition 3.52.** *There exists* $c$ *independent of* $t$ *such that for all* $a \in \Omega^2(N_t)$, $a \perp \mathcal{K}$ *we have*

$$||a||_{\mathfrak{X}_t} \leq c\, ||\Delta a||_{\mathfrak{Y}_t}. \tag{3.53}$$

The proof of this proposition will extend over the rest of the section. This linear estimate perpendicular to the approximate kernel is one thing we need. The following proposition states that by restricting to the orthogonal complement of $\mathcal{K}$ we are not forgetting about any important 2-forms — the image of the Laplacian remains the same when restricted to this orthogonal complement.

**Proposition 3.54.** *The operator*

$$\Delta : \mathcal{K}^\perp \longrightarrow \operatorname{Im} \Delta$$

*is surjective, where* $\operatorname{Im}\Delta$ *denotes the image of the Laplacian on all of* $\Omega^2(N_t)$.

*Proof.* **Step 1**: Show that the $L^2$-orthogonal projection $q : \operatorname{Ker} \Delta_{N_t} \longrightarrow \mathcal{K}$ is an isomorphism.

Assume there exists $0 \neq a \in \Omega^2(N_t)$ with $\Delta a = 0$ such that $q(a) = 0$, i.e. $a \perp \mathcal{K}$. Then $\Delta a \neq 0$ by Proposition 3.52, which is a contradiction. Now note $\dim(\operatorname{Ker} \Delta_{N_t}) = b^0(L) + b^2(T^7/\Gamma) = 12 + k$, which is proved using the Künneth formula (see [JK21, Proposition 6.1]). By construction, $\dim(\mathcal{K}) = 12 + k$, so $q$ is a surjective linear map between vector spaces of the same dimension, and therefore injective.

**Step 2**: Check $\operatorname{Im}(\Delta|_{\mathcal{K}^\perp}) = \operatorname{Im}\Delta$.

It suffices to check that $\operatorname{Im}\Delta \subset \operatorname{Im}(\Delta|_{\mathcal{K}^\perp})$. Let $y \in \operatorname{Im}\Delta$, and $\Delta x = y$. Denote the $L^2$-orthogonal projection onto $\mathcal{K}$ by $\mathrm{proj}_{\mathcal{K}}$. Let

$$z := q^{-1}(\mathrm{proj}_{\mathcal{K}}(-x)).$$

Then $\Delta(x+z) = y$, and $\mathrm{proj}_{\mathcal{K}}(x+z) = 0$ because of $\mathrm{proj}_{\mathcal{K}} \circ q^{-1} = \mathrm{Id}$, i.e. $x+z \perp \mathcal{K}$ which completes the proof. □

**Comparison with the Laplacian on $L$** The embedding $\iota_t : \Omega^0(L) \to \Omega^2(N_t)$ is defined using a cut-off of $\overline{\nu} \in \Omega^2(X_{\mathrm{EH}})$. If not for this cut-off, we would have that $\Delta\iota_t = \iota_t\Delta$, where we use the symbol $\Delta$ to denote the Laplacian on $N_t$ as well as the Laplacian on $L$. In our actual situation, we still have that $\Delta$ and $\iota_t$ nearly commute, and that is the content of the following proposition.

**Proposition 3.55.** *For any $\beta \leq 0$ there exists $c > 0$ independent of $t$ such that for all $g \in \Omega^0(L)$ we have*

$$\|(\Delta\iota_t - \iota_t\Delta)g\|_{C^{0,\alpha}_{\beta-2;t}} \leq ct^2 \|g\|_{C^{2,\alpha}}. \tag{3.56}$$

*Proof.* Define the map $\widetilde{\iota_t} : \Omega^0(L) \to \Omega^2(T^3 \times X_{\mathrm{EH}})$ via $\widetilde{\iota_t}(g) = p_L^* g \cdot p_{X_{\mathrm{EH}}}^* \overline{\nu}$, where $\overline{\nu} \in \Omega^2(X_{\mathrm{EH}})$ is harmonic and has unit $L^2$-norm with respect to $g_{X_{\mathrm{EH}}}$. Then

$$(\Delta\widetilde{\iota_t} - \widetilde{\iota_t}\Delta)g = 0. \tag{3.57}$$

We aim to estimate

$$(\Delta\iota_t - \iota_t\Delta)g = \underbrace{(\Delta\iota_t - \Delta\widetilde{\iota_t})g}_{=:I} + \underbrace{(\Delta\widetilde{\iota_t} - \widetilde{\iota_t}\Delta)g}_{=:II} + \underbrace{(\widetilde{\iota_t}\Delta - \iota_t\Delta)g}_{=:III}.$$

We begin by estimating I, where it will be convenient to estimate on two regions separately:

$$\begin{aligned} \Omega_1 &:= \{x \in L \times X_{\mathrm{EH}} : \check{r}(x) \leq t^{-1}\zeta/8\}, \\ \Omega_2 &:= \{x \in L \times X_{\mathrm{EH}} : t^{-1}\zeta/8\} \leq \check{r}(x) \leq t^{-1}\zeta/4\}. \end{aligned} \tag{3.58}$$

Then

$$
\begin{aligned}
||I||_{C^{0,\alpha}_{\beta-2;t}} &\leq ||(\iota_t - \widetilde{\iota}_t)g||_{C^{2,\alpha}_{\beta;t}} \\
&= \left\|p_L^* g \cdot p_{X_{\mathrm{EH}}}^*(\nu - \overline{\nu})\right\|_{C^{2,\alpha}_{\beta;t}} \\
&\leq \left\|p_L^* g \cdot p_{X_{\mathrm{EH}}}^*(\nu - \overline{\nu})\right\|_{C^{2,\alpha}_{\beta;t}(\Omega_1)} + \left\|p_L^* g \cdot \chi_t p_{X_{\mathrm{EH}}}^*(\nu - \overline{\nu})\right\|_{C^{2,\alpha}_{\beta;t}(\Omega_2)}
\end{aligned}
$$

We will estimate the two summands separately. The first summand is defined on the region $\Omega_1 = \{x \in L \times X_{\mathrm{EH}} : \check{r}(x) \leq t^{-1}\zeta/8\}$ where neither $\overline{\nu}$ nor $\nu$ is cut off. We have that

$$
|\nu(x) - \overline{\nu}(x)|_{t^2 g_{X_{\mathrm{EH}}}} \leq ct^2 \text{ for } x \in X_{\mathrm{EH}} \text{ with } \check{r}(x) \leq t^{-1}\zeta/8 \tag{3.59}
$$

for the following reason: $\langle \overline{\nu}, \overline{\nu} \rangle_{L^2, t^2 g_{X_{\mathrm{EH}}}} = 1$ by definition, thus

$$
\begin{aligned}
\langle \chi_t \overline{\nu}, \chi_t \overline{\nu} \rangle_{L^2, t^2 g_{X_{\mathrm{EH}}}} &\geq \langle \overline{\nu}, \overline{\nu} \rangle_{L^2, t^2 g_{X_{\mathrm{EH}}}} - \int_{\{x \in X_{\mathrm{EH}} : \check{r}(x) \geq \zeta t^{-1}/8\}} |\overline{\nu}|^2_{t^2 g_{X_{\mathrm{EH}}}} \mathrm{vol}_{t^2 g_{X_{\mathrm{EH}}}} \\
&\geq 1 - \int_{\zeta t^{-1}/8}^{\infty} (1+r)^{-8} r^3 \, \mathrm{d}r \geq 1 - ct^4.
\end{aligned}
$$

If $\check{r}(x) \leq t^{-1}\zeta/8$ we have that $\nu(x) = \overline{\nu}(x) / \langle \chi_t \overline{\nu}, \chi_t \overline{\nu} \rangle_{L^2, t^2 g_{X_{\mathrm{EH}}}}$ because the cut-off is applied where $\check{r}(x) > t^{-1}\zeta/8$. This implies, at the point $x$,

$$
|\nu - \overline{\nu}|_{t^2 g_{X_{\mathrm{EH}}}} \leq \left| \overline{\nu} \left( 1 - \frac{1}{\langle \chi_t \overline{\nu}, \chi_t \overline{\nu} \rangle_{L^2, t^2 g_{X_{\mathrm{EH}}}}} \right) \right|_{t^2 g_{X_{\mathrm{EH}}}} \leq \left| \overline{\nu} \cdot \frac{t^4}{1 - t^4} \right|_{t^2 g_{X_{\mathrm{EH}}}} \leq t^{-2} \left| \overline{\nu} \cdot \frac{t^4}{1 - t^4} \right|_{g_{X_{\mathrm{EH}}}} \leq ct^2.
$$

Using this for our estimate of the first summand of I, we obtain:

$$
\left\|p_L^* g \cdot p_{X_{\mathrm{EH}}}^*(\nu - \overline{\nu})\right\|_{C^{2,\alpha}_{\beta;t}(\Omega_1)} \leq t^2 \left\|p_L^* g\right\|_{C^{2,\alpha}_{\beta;t}} \leq ct^2 \, ||g||_{C^{2,\alpha}}.
$$

For the second summand we get:

$$
\begin{aligned}
&\left\|p_L^* g \cdot \chi_t p_{X_{\mathrm{EH}}}^* (\nu - \overline{\nu})\right\|_{C^{2,\alpha}_{\beta;t}(\Omega_2)} \\
\leq{}& \left\|p_{T^3}^* g\right\|_{C^{2,\alpha}_{0;t}} \left\|\chi_t p_{X_{\mathrm{EH}}}^* (\nu - \overline{\nu})\right\|_{C^{2,\alpha}_{\beta;t}(\Omega_2)} \\
\leq{}& \left\|p_{T^3}^* g\right\|_{C^{2,\alpha}_{0;t}} \|\chi_t\|_{C^{2,\alpha}_{0;t}} \cdot \|1\|_{C^{2,\alpha}_{\beta+4;t}(\Omega_2)} \left(\|\nu\|_{C^{2,\alpha}_{-4;t}(\Omega_2)} + \|\overline{\nu}\|_{C^{2,\alpha}_{-4;t}(\Omega_2)}\right) \\
\leq{}& ct^2 \|g\|_{C^{2,\alpha}},
\end{aligned}
$$

where in the last step we used $\|1\|_{C^{2,\alpha}_{\beta+4,0;t}(\Omega_2)} \leq c$, which holds because far away from $L$, the weight function $w_{\beta+4,0;t}$ is uniformly bounded. We also used

$$
|\overline{\nu}|_{t^2 g_{X_{\mathrm{EH}}}} = t^{-2} |\overline{\nu}|_{g_{X_{\mathrm{EH}}}} \leq ct^{-2}(1+\check{r})^{-4} \leq ct^2 (t + t\check{r})^{-4} \leq ct^2 \text{ on } \Omega_2. \tag{3.60}
$$

Together with Eq. (3.59) this shows that $|\nu|_{t^2 g_{X_{\mathrm{EH}}}} \leq ct^2$ on $\Omega_2$.

Altogether $\|I\|_{C^{0,\alpha}_{\beta-2;t}} \leq ct^2 \|g\|_{C^{2,\alpha}}$. Furthermore, $II = 0$ because of Eq. (3.57). Lastly, III is estimated like I, which shows the claim. □

The goal of this section is to prove an estimate for the operator norm of the inverse of the Laplacian with respect to the norms $\|\cdot\|_{\mathfrak{X}_t}$ and $\|\cdot\|_{\mathfrak{Y}_t}$. The purpose of these norms is to essentially split the problem into an estimate on $\operatorname{Im} \pi_t$ and remainder. The following proposition contains the estimate on $\operatorname{Im} \pi_t$.

**Proposition 3.61.** *There exists $c > 0$ independent of $t$ such that for $t$ small enough and for all $g \in \Omega^0(L)$ satisfying $g \perp \operatorname{Ker} \Delta_L$ we have that*

$$
\|g\|_{C^{2,\alpha}} \leq c \|\pi_t \Delta \iota_t g\|_{C^{0,\alpha}}. \tag{3.62}
$$

*Proof.* We have

$$\begin{aligned}
||g||_{C^{2,\alpha}} &\leq c\,||\Delta g||_{C^{0,\alpha}} \\
&= c\,||\pi_t \iota_t \Delta g||_{C^{0,\alpha}} \\
&\leq c\,||\pi_t \Delta \iota_t g||_{C^{0,\alpha}} + c\,||\pi_t \Delta \iota_t g - \pi_t \iota_t \Delta g||_{C^{0,\alpha}} \\
&\leq c\,||\pi_t \Delta \iota_t g||_{C^{0,\alpha}} + ct^{2-\alpha}\,||g||_{C^{2,\alpha}},
\end{aligned}$$

where we used elliptic regularity for the operator $\Delta$ on $L$ in the first step, and Propositions 3.48 and 3.55 in the last step. At this point, the last summand $ct^{2-\alpha}\,||g||_{C^{2,\alpha}}$ can be absorbed into the left hand side for $t$ small enough. □

**The model operator on $\mathbb{R}^3 \times X_{\mathrm{EH}}$**

*Definition* 3.63. For $j \in \{1, \ldots, 12\}$ let $C'_j$ be a connected component of $\hat{U}$, but made slightly smaller, explicitly

$$C'_j := C_j \cap \{(x_h, x_v) \in L \times X_{\mathrm{EH}} : \check{r}(x_v) \leq t^{-1}\zeta/4\}.$$

For $\beta \in \mathbb{R}$ let

$$\begin{aligned}
s_{j,\beta,t} : \Omega^2(N_t) &\longrightarrow \Omega^2(\mathbb{R}^3 \times \{x \in X_{\mathrm{EH}} : \check{r}(x) \leq t^{-1}\zeta/4\}) \\
a &\longmapsto t^{-\beta-2}(p \circ (\cdot t), \mathrm{Id})^* \left(a|_{C'_j}\right),
\end{aligned}$$

where $p : \mathbb{R}^3 \longrightarrow T^3$ denotes the quotient map.

Then:

**Lemma 3.64.** *For $j \in \{1, \ldots, 12\}$, $\beta \in \mathbb{R}$ we have that for all $a \in \Omega^2(\mathbb{R}^3 \times X_{EH})$ we have*

$$\left\|s_{j,\beta,t} a\right\|_{C^{k,\alpha}_{\beta}} = ||a||_{C^{k,\alpha}_{\beta;t}(C'_j)}, \text{ and}$$

$$\left(s_{j,\beta-2,t} \Delta_{N_t} a - \Delta_{g_{\mathbb{R}^3} \oplus g_{(1)}} s_{j,\beta,t} a\right)|_{C'_j} = 0.$$

*Here $\Delta_{g_{\mathbb{R}^3} \oplus g_{(1)}}$ denotes the Laplacian on $\mathbb{R}^3 \times X_{EH}$ with respect to the metric $g_{\mathbb{R}^3} \oplus g_{(1)}$.*

*Proof.* The map $((\cdot t)\circ p, \mathrm{Id}) : C'_j \to \mathbb{R}^3 \times \{x \in X_{\mathrm{EH}} : \check{r}(x) \le t^{-1}\zeta/4\}$ pulls back the metric $t^2(g_{\mathbb{R}^3} \oplus g_{(1)})$ to the metric induced by $\varphi^t$. The extra factor $t^{-\beta-2}$ cancels out the factor $t^2$ when changing the metric from $t^2(g_{\mathbb{R}^3} \oplus g_{(1)})$ to $g_{\mathbb{R}^3} \oplus g_{(1)}$ on 2-forms and cancels out the factor $t^\beta$ coming from the definition of $||\cdot||_{C^{k,\alpha}_{\beta;t}}$. ◻

**Estimate of $\rho_t a$** In Proposition 3.61 we essentially proved an estimate for the inverse of the Laplacian on $\operatorname{Im}\pi_t$. In order to get an estimate with respect to $||\cdot||_{\mathfrak{X}_t}$ and $||\cdot||_{\mathfrak{Y}_t}$ we need to estimate the inverse of the Laplacian on $\operatorname{Im}\rho_t$. Recall the projection $\pi_t$ onto the fibrewise harmonic part from Eq. (3.42) and its complement $\rho_t$. The two operators satisfy $\pi_t\rho_t = 0$, so the following proposition implies an estimate for the inverse of the Laplacian for elements $a \in \operatorname{Im}\rho_t \subset \Omega^2(N_t)$.

**Proposition 3.65.** *Write $\mathcal{K}' := \{(1-\chi_t)a : a \in \operatorname{Ker}\Delta_{T^7/\Gamma}\} \subset \Omega^2(N_t)$. Then there exists $c > 0$ independent of $t$ such that for $a \in \Omega^2(N_t)$ satisfying $a \perp \mathcal{K}'$ we have*

$$||a||_{C^{2,\alpha}_{\beta;t}} \le c\left(||\Delta a||_{C^{0,\alpha}_{\beta-2;t}} + ||\overline{\pi}_t a||_{L^\infty_{\beta;t}}\right). \tag{3.66}$$

*Proof.* The Schauder estimate

$$||a||_{C^{2,\alpha}_{\beta;t}} \le c\left(||\Delta a||_{C^{0,\alpha}_{\beta-2;t}} + ||a||_{L^\infty_{\beta;t}}\right) \tag{3.67}$$

can be derived as in [Wal17, Proposition 8.15]. It then suffices to show that there exists $c$ such that

$$||a||_{L^\infty_{\beta;t}} \le c\left(||\Delta a||_{C^{0,\alpha}_{\beta-2;t}} + ||\overline{\pi}_t a||_{L^\infty_{\beta;t}}\right). \tag{3.68}$$

Assume Eq. (3.68) is false, then there exist $t_i \to 0$, $a_i \in \Omega^2(N_{t_i})$ satisfying $a_i \perp \mathcal{K}'$, and $x_i \in N_{t_i}$ such that

$$||a||_{C^{2,\alpha}_{\beta;t_i}} \le c, \left|w_{\beta;t_i}(x_i)a_i(x_i)\right| = 1, \text{ and } ||\Delta a_i||_{C^{0,\alpha}_{\beta-2;t_i}} \to 0, \left\|\overline{\pi}_{t_i} a_i\right\|_{L^\infty_{\beta;t_i}} \to 0. \tag{3.69}$$

Here, we got $||a||_{C^{2,\alpha}_{\beta;t_i}} \le c$ from Eq. (3.67). Without loss of generality we can assume to be in one of three following cases, and we will arrive at a contradiction in each of them.

**Case 1**: the sequence $x_i$ concentrates on one ALE space, i.e. $t_i^{-1} r_{t_i}(x_i) \to c < \infty$ (see Fig. 2).

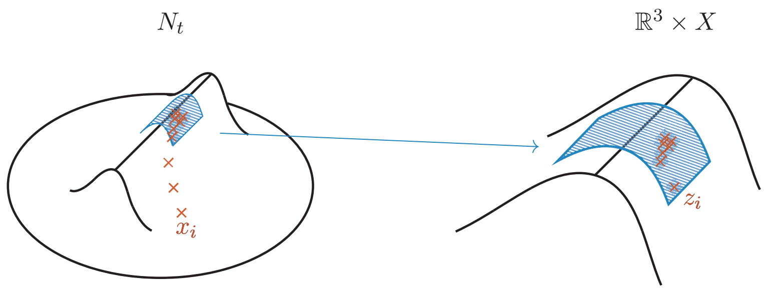


Figure 2: Blowup analysis near the associative is reduced to the analysis of the Laplacian on $\mathbb{R}^3 \times X_{\mathrm{EH}}$.

By passing to a subsequence and translating in the $\mathbb{R}^3$-direction if necessary, we can assume that $x_i$ concentrates near one fixed connected component of $L$. Let $C_j \subset L \times X_{\mathrm{EH}}$ be the connected component $\hat{U}$ containing an accumulation point of the sequence $x_i$. Define $\widetilde{a}_i := s_{j,\beta,t} a_i \in \Omega^2(\mathbb{R}^3 \times \{x \in X_{\mathrm{EH}} : \check{r}(x) \leq t_i^{-1}\zeta/4\})$ and let $\widetilde{x}_i$ be a lift from $C_j$ to $\mathbb{R}^3 \times X_{\mathrm{EH}}$. The new 2-form $\widetilde{a}_i$ then satisfies

$$||\widetilde{a}_i||_{C^{2,\alpha}_\beta} \leq c, \; (1+\check{r}(\widetilde{x}_i))^{-\beta} \, |\widetilde{a}_i(\widetilde{x}_i)| \geq c, \text{ and } ||\Delta \widetilde{a}_i||_{C^{0,\alpha}_{\beta-2}} \to 0,$$

which follows from Lemma 3.64. Now the weight function no longer has $t_i$ in it and distances and tensors are measured using the metric $g_{\mathbb{R}^3} \oplus g_{(1)}$.

By the assumption of case 1, we have $\check{r}(\widetilde{x}_i) \to c < \infty$. By passing to a subsequence we can assume that $\widetilde{x}_i$ converges, so write $x^* := \lim_{i \to \infty} \widetilde{x}_i \in \mathbb{R}^3 \times X_{\mathrm{EH}}$. Using the Arzelà-Ascoli theorem and a diagonal argument, we can extract a limit $a^* \in \Omega^2(\mathbb{R}^3 \times X_{\mathrm{EH}})$ of the sequence $\widetilde{a}_i$ satisfying:

$$||a^*||_{L^\infty_\beta} \leq c, \text{ and} \tag{3.70}$$

$$\Delta_{g_{\mathbb{R}^3} \oplus g_{(1)}} a^* = 0, \text{ and} \tag{3.71}$$

$$(1+\check{r}(x^*))^{-\beta} \, |a^*(x^*)| > c. \tag{3.72}$$

By Corollary 3.39 (applied to the case $\mathbb{R}^3 \times X_{\mathrm{EH}}$), we have that $a^*$ is independent of the $\mathbb{R}^3$-

direction. By Proposition 3.21, the only harmonic forms on $X_{\mathrm{EH}}$ that decay like $\check{r}^{\beta}$ are multiples of $\nu_1$. Thus $a^*$ is the pullback of a multiple of $\nu_1$ under the projection $p_{X_{\mathrm{EH}}} : \mathbb{R}^3 \times X_{\mathrm{EH}} \longrightarrow X_{\mathrm{EH}}$.

Because $\left\|\overline{\pi}_{t_i} a_i\right\|_{L^\infty_{\beta;t_i}} \longrightarrow 0$, we have that $a^*$ is perpendicular to $\overline{\nu}$ on every $\{y\} \times X_{\mathrm{EH}} \subset \mathbb{R}^3 \times X_{\mathrm{EH}}$. Here is how to see this in detail: let $y \in L$, then we calculate on $\{y\} \times X_{\mathrm{EH}}$:

$$\langle a^*, \overline{\nu} \rangle = \langle a^*, \overline{\nu} - \chi_t \nu \rangle + \langle a^* - \widetilde{a}_i, \chi_t \nu \rangle + \langle \widetilde{a}_i, \chi_t \nu \rangle = I + II + III. \tag{3.73}$$

Here,

$$|I| \leq \left| \langle a^*, \overline{\nu} - \chi_t \nu \rangle_{\{x \in X_{\mathrm{EH}} : \check{r}(x) \leq t^{-1}\zeta/8\}} \right| + \left| \langle a^*, \overline{\nu} - \chi_t \nu \rangle_{\{x \in X_{\mathrm{EH}} : \check{r}(x) \geq t^{-1}\zeta/8\}} \right|,$$

where we have for the first summand

$$\begin{aligned} \left| \langle a^*, \overline{\nu} - \chi_t \nu \rangle_{\{x \in X_{\mathrm{EH}} : \check{r}(x) \leq t^{-1}\zeta/8\}} \right| &\leq \int_0^{t^{-1}\zeta/8} |a^*|_{g_{(1)}} \cdot |\overline{\nu} - \chi_t \nu|_{g_{(1)}} r^3 \,\mathrm{d}r \\ &\leq c \int_0^{t^{-1}\zeta/8} r^{\beta} t^4 r^3 \,\mathrm{d}r \leq c t^{-\beta} \longrightarrow 0. \end{aligned}$$

Here we used Eq. (3.70) and Eq. (3.59) (after changing from $|\cdot|_{t^2 g_{X_{\mathrm{EH}}}}$ to $|\cdot|_{g_{X_{\mathrm{EH}}}}$) in the second step. For the second summand we find

$$\left| \langle a^*, \overline{\nu} - \chi_t \nu \rangle_{\{x \in X_{\mathrm{EH}} : \check{r}(x) \geq t^{-1}\zeta/8\}} \right| \leq c \int_{\zeta/8t^{-1}}^{\infty} r^{\beta} r^{-4} r^3 \,\mathrm{d}r \leq c t^{-\beta} \longrightarrow 0,$$

where we used $\overline{\nu} = O(\check{r}^{-4})$ and $\nu = O(\check{r}^{-4})$ in the first step.

In order to estimate $II$, let $l > 0$. Then

$$|II| \leq \left| \langle a^* - \widetilde{a}_i, \chi_t \nu \rangle_{\{x \in X_{\mathrm{EH}} : \check{r}(x) \geq l\}} \right| + \left| \langle a^* - \widetilde{a}_i, \chi_t \nu \rangle_{\{x \in X_{\mathrm{EH}} : \check{r}(x) \leq l\}} \right|,$$

and we find for the first summand

$$\left| \langle a^* - \widetilde{a}_i, \chi_t \nu \rangle_{\{x \in X_{\mathrm{EH}} : \check{r}(x) \geq l\}} \right| \leq c \left( ||a^*||_{L^\infty_\beta} + ||\widetilde{a}_i||_{L^\infty_\beta} \right) \int_l^{\infty} r^{\beta-4+3} \,\mathrm{d}r \leq c l^{\beta}$$

for a constant $c$ independent of $l$. For the second summand we have

$$
\begin{aligned}
\left|\langle a^* - \widetilde{a}_i, \chi_t \nu\rangle_{\{x\in X_{\mathrm{EH}}:\check{r}(x)\le l\}}\right| &\le \|a^* - \widetilde{a}_i\|_{L^\infty_\beta(\{x\in X_{\mathrm{EH}}:\check{r}(x)\le l\})} \cdot \int_0^l r^{\beta-4+3}\,\mathrm{d}r \\
&\le c\,\|a^* - \widetilde{a}_i\|_{L^\infty_\beta(\{x\in X_{\mathrm{EH}}:\check{r}(x)\le l\})} \longrightarrow 0
\end{aligned}
$$

as $i \longrightarrow \infty$ by definition of $a^*$. Last,

$$
|III| = t^{-2-\beta}|(\pi_t a_i)(y)| = t^{-2-\beta}|(\pi_t \iota_t \pi_t a_i)(y)| \le c\,\|\overline{\pi}_t a_i\|_{L^\infty_{\beta;t}} \longrightarrow 0,
$$

where we used Proposition 3.48 for the estimate.

Altogether we see that, by taking $\lim_{i\to\infty}$ in Eq. (3.73), we have that $\langle a^*, \overline{\nu}\rangle \le cl^\beta$, where the constant $c$ was independent of $l$. This is true for any $l > 0$, therefore $\langle a^*, \overline{\nu}\rangle = 0$. But this is a contradiction to Eq. (3.72).

**Case 2**: the sequence $x_i$ concentrates on the regular part, i.e. $r_{t_i}(x_i) \longrightarrow c > 0$ (see Fig. 3).

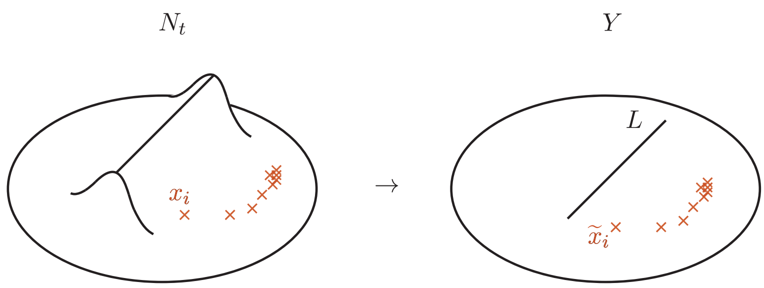


Figure 3: Blowup analysis away from the associative is reduced to the analysis of the Laplacian on $T^7/\Gamma$.

Using the Arzelà-Ascoli theorem and a diagonal argument, we extract a limit $a^* \in \Omega^2(T^7/\Gamma\backslash L)$. Denote, furthermore, $\lim_{i\to\infty} x_i = x^*$. We have $|a^*| < c \cdot d(\cdot, L)^\beta$, so we have that $a^*$ is a well-defined distribution on $M/\langle\iota\rangle$ acting on $L^2$-sections because $\beta > -2$. We also have $\Delta a^* = 0$, so $a^*$ is smooth by elliptic regularity, e.g. [Fol95, Theorem 6.33].

Furthermore,

$$
\langle a^*, (1-\chi(2d(\cdot,L)))\cdot \alpha_i\rangle_{T^7/\Gamma} = \lim_{i\to\infty}\langle a_i, (1-\chi_t(r_t))\cdot \pi^*\alpha_i\rangle_{N_{t_i}} = 0. \tag{3.74}
$$

By the unique continuation property for elliptic PDEs, the inner product

$$\langle \cdot , (1-\chi) \circ (2d(\cdot, L)) \cdot \rangle$$

is non-degenerate on harmonic forms. The 2-form $a^*$ is a harmonic form that is orthogonal to all harmonic forms with respect to this inner product, therefore $a^* = 0$. But this contradicts $a^*(x^*) > c$.

**Case 3**: the sequence $x_i$ concentrates on the neck region, i.e. $\check{r}(x_i) \to \infty$, but $r_t(x_i) \to 0$ (see Fig. 4).

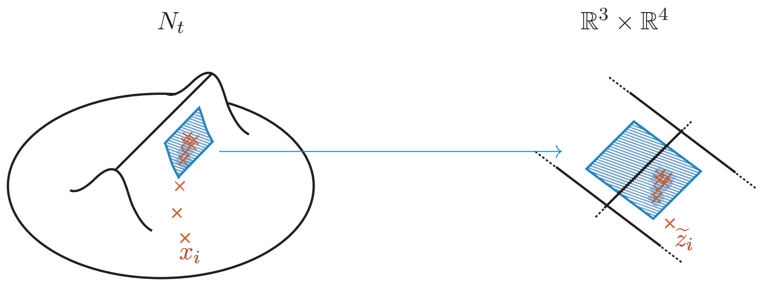


Figure 4: Blowup analysis in the neck region is reduced to the analysis of the Laplacian on $\mathbb{R}^3 \times \mathbb{R}^4$.

Define $\widetilde{a}_i \in \Omega^2(\mathbb{R}^3 \times X_{\mathrm{EH}})$ and $\widetilde{x}_i \in \mathbb{R}^3 \times X_{\mathrm{EH}}$ as in case 1. In this case, we have that $|\rho(\widetilde{x}_i)| \to \infty$. In order to be able to obtain a limit of this sequence, let $R_i \to \infty$ be a sequence such that $R_i/|\rho(\widetilde{x}_i)| \to 0$. Cutting out the exceptional locus of the Eguchi-Hanson space, we can consider $\{(x_h, x_v) \in \mathbb{R}^3 \times X_{\mathrm{EH}} : R_i \leq |\rho|(x_v) \leq \zeta t_i^{-1}\}$ as a subset of $\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}$. On $\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}$, we have the rescaling map $(\cdot\, |\rho(\widetilde{x}_i)|)$.

We now define:

$$\begin{aligned} \widetilde{\widetilde{a}}_i &:= (\cdot\, |\rho(\widetilde{x}_i)|)^* \left( \widetilde{a}_i|_{\{R_i \leq |\rho| \leq \zeta t_i^{-1}\}} \right) \cdot |\rho(\widetilde{x}_i)|^{-2-\beta} \\ &\quad \in \Omega^2(\mathbb{R}^3 \times \{x \in X_{\mathrm{EH}} : R_i/|\rho(\widetilde{x}_i)| \leq |\rho(x)| \leq \zeta t_i^{-1}/|\rho(\widetilde{x}_i)|\}), \\ \widetilde{\widetilde{x}}_i &:= \widetilde{x}_i/|\rho(\widetilde{x}_i)|\,. \end{aligned} \tag{3.75}$$

This sequence satisfies

$$\left\|\widetilde{\widetilde{a}}_i\right\|_{C^{2,\alpha}_\beta} \leq c \text{ and } \left|\widetilde{\widetilde{a}}_i(\widetilde{\widetilde{x}}_i)\right| > c. \tag{3.76}$$

The data $\widetilde{\widetilde{a}}_i$ and $\widetilde{\widetilde{x}}_i$ are defined on (subsets of) $\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}$. We use the same symbols to denote their pullbacks under the quotient map $\mathbb{C}^2 \to \mathbb{C}^2/\{\pm 1\}$.

As before, we extract a $C^{2,\alpha/2}_{loc}$-limit $a^* \in \Omega^2(\mathbb{R}^3 \times \mathbb{R}^4 \setminus \{0\})$ satisfying

$$\Delta_{\mathbb{R}^7} a^* = 0, \text{ and } ||a^*||_{L^\infty_\beta(\mathbb{R}^3 \times \mathbb{R}^4)} \leq c.$$

We see as in case 2 that $a^*$ defines a distribution on all of $\mathbb{R}^7$, and is smooth by elliptic regularity on all of $\mathbb{R}^7$.

We also get an $L^\infty$-bound for $a^*$ as follows: away from $\mathbb{R}^3 \times \{0\}$, this is given by Eq. (3.76). To see that $a^*$ does not blow up in the $\mathbb{R}^3$-direction near $\mathbb{R}^3 \times \{0\}$, consider any $y \in \mathbb{R}^3 \times \{0\}$. Let $1 < p < -4/\beta$, then $||a^*||_{L^p(B_1(y))} \leq c$, independent of $y$, by Eq. (3.76). So, by elliptic regularity $||a^*||_{L^p_m(B_1(y))} \leq c$ for any $m \in \mathbb{N}$, and by the Sobolev embedding we have $||a^*||_{L^\infty} \leq c$, where all of these estimates were independent of $y$.

By Corollary 3.39 (applied to $\mathbb{R}^3 \times \mathbb{R}^4$), $a^*$ is constant in the $\mathbb{R}^3$ direction. The limit $a^*$ is therefore the pullback of a harmonic, decaying form of $\mathbb{R}^4$, and must thus vanish, which is a contradiction to the second part of Eq. (3.76). □

**Cross-term estimates** We have now established uniform estimates for the inverse of $\Delta$ on $\operatorname{Im} \pi_t$ and $\operatorname{Im} \rho_t$. As it stands, it could happen that the operator norm of $\rho_t \Delta \overline{\pi}_t$ or $\pi_t \Delta \rho_t$ is very big. It will turn out in our proof of Proposition 3.52 that in such a case one would be unable to deduce anything about the inverse of the operator norm of $\Delta$ with respect to $||\cdot||_{\mathfrak{X}_t}$ and $||\cdot||_{\mathfrak{Y}_t}$. Fortunately, it turns out that the operator norms of $\rho_t \Delta \iota_t$ (and therefore $\rho_t \Delta \overline{\pi}_t$, because $\overline{\pi}_t = \iota_t \pi_t$) and $\pi_t \Delta \rho_t$ are small, which is the content of the following proposition.

**Proposition 3.77.** *There exists $c > 0$ independent of $t$ such that for all $g \in \Omega^0(L)$ and for all*

$a \in \Omega^2(N_t)$ *we have*

$$||\rho_t \Delta \iota_t g||_{C^{0,\alpha}_{\beta;t}} \leq c t^{2-\alpha} \, ||g||_{C^{2,\alpha}} \quad \text{if } \beta < 0, \tag{3.78}$$

$$||\pi_t \Delta \rho_t a||_{C^{0,\alpha}} \leq c t^{2+2\beta-2\alpha} \, ||\rho_t a||_{C^{2,\alpha}_{\beta;t}} \quad \text{if } -2 < \beta < 0. \tag{3.79}$$

*Proof.* We first prove Eq. (3.78). We have $\rho_t \iota_t = 0$ and therefore

$$\begin{aligned}
||\rho_t \Delta \iota_t g||_{C^{0,\alpha}_{\beta;t}} &= ||\rho_t (\Delta \iota_t g - \iota_t \Delta g)||_{C^{0,\alpha}_{\beta;t}} \\
&\leq ||\Delta \iota_t g - \iota_t \Delta g||_{C^{0,\alpha}_{\beta;t}} + ||\iota_t \pi_t (\Delta \iota_t g - \iota_t \Delta g)||_{C^{0,\alpha}_{\beta;t}} \\
&\leq ||\Delta \iota_t g - \iota_t \Delta g||_{C^{0,\alpha}_{\beta;t}} + c t^{-2-\beta} \, ||\pi_t (\Delta \iota_t g - \iota_t \Delta g)||_{C^{0,\alpha}} \\
&\leq ||\Delta \iota_t g - \iota_t \Delta g||_{C^{0,\alpha}_{\beta;t}} + c t^{-\alpha} \, ||\Delta \iota_t g - \iota_t \Delta g||_{C^{0,\alpha}_{\beta;t}} \\
&\leq c t^{2-\alpha} \, ||g||_{C^{2,\alpha}} ,
\end{aligned}$$

where we used Proposition 3.45 in the third step, Proposition 3.48 in the fourth step, and Proposition 3.55 in the last step.

Now to prove Eq. (3.79): assume without loss of generality that $a = \rho_t a$. Define

$$\begin{aligned}
\widetilde{\pi}_t : \Omega^2(T^3 \times X_{\mathrm{EH}}) &\to \Omega^0(L) \\
(\widetilde{\pi}_t a)(x) &:= \langle a, \overline{\nu} \rangle_{t^2 g_{X_{\mathrm{EH}}}} .
\end{aligned}$$

The difference between $\widetilde{\pi}_t$ and $\pi_t$ is that they use $\overline{\nu}$ and $\chi_t \nu$ in their definition, respectively: $\overline{\nu}$ is not cut off, $\chi_t \nu$ is, and both are rescaled to have unit norm. It suffices to prove the claim for $a \in \Omega^2(N_t)$ which is supported near $L$. We can view such $a$ as an element in $\Omega^2(T^3 \times X_{\mathrm{EH}})$ and apply $\widetilde{\pi}_t$ to it. Also define $\widetilde{\iota}_t : \Omega^0(L) \to \Omega^2(T^3 \times X_{\mathrm{EH}})$ as $\widetilde{\iota}_t(g) = p^*_{T^3} \cdot p^*_{X_{\mathrm{EH}}} \overline{\nu}$. Then $\widetilde{\pi}_t \widetilde{\iota}_t = \mathrm{Id}$ and we also define $\widetilde{\rho}_t := 1 - \widetilde{\iota}_t \widetilde{\pi}_t$.

We have $\widetilde{\pi}_t \Delta = \Delta \widetilde{\pi}_t$, thus $\widetilde{\pi}_t a = 0 \Rightarrow \widetilde{\pi}_t \Delta a = 0$, and therefore $\widetilde{\pi}_t \Delta \widetilde{\rho}_t = 0$. Hence

$$\pi_t \Delta \rho_t a = \underbrace{(\pi_t - \widetilde{\pi}_t) \Delta \rho_t a}_{=:I} + \underbrace{\widetilde{\pi}_t \Delta (\rho_t - (1 - \iota_t \widetilde{\pi}_t)) a}_{=:II} + \underbrace{\widetilde{\pi}_t \Delta ((1 - \iota_t \widetilde{\pi}_t) - \widetilde{\rho}_t) a}_{=:III} .$$

We first estimate I:

$$\langle \Delta\rho_t a, \overline{v} - \chi_t v\rangle_{L^2, t^2 g_{X_{\mathrm{EH}}}} \leq \underbrace{ct^{4+\beta}\int_0^{t^{-1}\zeta/8}\left(||\Delta\rho_t a||_{C^{0,\alpha}_{\beta-2;t}}(1+r)^{-2+\beta}\right) r^3\,\mathrm{d}r}_{\leq ct^{2+\beta}||\rho_t a||_{C^{2,\alpha}_{\beta;t}} \text{ if } -2\leq\beta\leq 0}$$
$$+ \underbrace{ct^{\beta}\int_{t^{-1}\zeta/8}^{\infty} ||\rho_t a||_{C^{2,\alpha}_{\beta;t}}(1+r)^{-2+\beta-4} r^3\,\mathrm{d}r}_{\leq ct^2||\rho_t a||_{C^{2,\alpha}_{\beta;t}}}.$$

Here we applied Eq. (3.59) on the region $\{x \in X_{\mathrm{EH}} : \check{r}(x) \leq \zeta t^{-1}/8\}$ and we used

$$|\overline{v} - \chi_t v|_{t^2 g_{X_{\mathrm{EH}}}} \leq |\overline{v}|_{t^2 g_{X_{\mathrm{EH}}}} + |\chi_t v|_{t^2 g_{X_{\mathrm{EH}}}} \leq c(t+\check{r}t)^{-4}t^2$$

on the region $\{x \in X_{\mathrm{EH}} : \check{r}(x) \geq \zeta t^{-1}/8\}$. Thus

$$||(\pi_t - \widetilde{\pi}_t)\Delta\rho_t a||_{L^\infty} \leq ct^{2+\beta}\,||\rho_t a||_{C^{2,\alpha}_{\beta;t}}$$

and the $C^{0,\alpha}$-estimate follows analogously.

For estimating II we need the estimate

$$||\widetilde{\pi}_t a||_{C^{k,\alpha}} \leq t^{2+\beta-\alpha-k}\,||a||_{C^{k,\alpha}_{\beta;t}}. \tag{3.80}$$

which is proved like Proposition 3.48. Then

$$\begin{aligned}
||\widetilde{\pi}_t\Delta(\rho_t - (1-\iota_t\widetilde{\pi}_t))a||_{C^{0,\alpha}} &= ||\widetilde{\pi}_t\Delta(\iota_t\pi_t - \iota_t\widetilde{\pi}_t)a||_{C^{0,\alpha}} \\
&\leq ct^{-\alpha}\,||\Delta\iota_t(\pi_t - \widetilde{\pi}_t)a||_{C^{0,\alpha}_{-2;t}} \\
&\leq ct^{-\alpha}\left(||\iota_t\Delta(\pi_t - \widetilde{\pi}_t)a||_{C^{0,\alpha}_{-2;t}} + t^2\,||(\pi_t - \widetilde{\pi}_t)a||_{C^{2,\alpha}}\right) \\
&\leq ct^{-\alpha}(1+t^2)\,||(\pi_t - \widetilde{\pi}_t)a||_{C^{2,\alpha}} \\
&\leq ct^{-\alpha}(1+t^2)t^2\,||a||_{C^{2,\alpha}_{\beta;t}} \\
&\leq ct^{2-\alpha}\,||\rho_t a||_{C^{2,\alpha}_{\beta;t}}
\end{aligned}$$

where in the first estimate we used Eq. (3.80), in the second estimate we used Proposition 3.55, in the third estimate we used the estimate for the operator norm of $\iota_t$ from Proposition 3.45, and in the fourth estimate we did the same calculation as when estimating I and we again used $-2 < \beta < 0$. In the last step we used the assumption that $a = \rho_t a$.

It remains to estimate III. We find

$$\begin{aligned}
||\widetilde{\pi}_t \Delta((1-\iota_t\widetilde{\pi}_t) - \widetilde{\rho}_t)a||_{C^{0,\alpha}} &= ||\widetilde{\pi}_t \Delta(\iota_t - \widetilde{\iota}_t)\widetilde{\pi}_t a||_{C^{0,\alpha}} \\
&\leq ct^{-\alpha+\beta}\,||\Delta(\iota_t - \widetilde{\iota}_t)\widetilde{\pi}_t a||_{C^{0,\alpha}_{\beta-2;t}} \\
&\leq ct^{-\alpha+\beta}\,||(\iota_t - \widetilde{\iota}_t)\Delta\widetilde{\pi}_t a||_{C^{0,\alpha}_{\beta-2;t}} + t^{2-\alpha+\beta}\,||\widetilde{\pi}_t a||_{C^{2,\alpha}}\,,
\end{aligned}$$

where we used Eq. (3.80) in the second step, and $\widetilde{\iota}_t\Delta = \Delta\widetilde{\iota}_t$ together with Proposition 3.55 in the third step. Here we find for the first summand

$$\begin{aligned}
ct^{-\alpha+\beta}\,||(\iota_t - \widetilde{\iota}_t)\Delta\widetilde{\pi}_t a||_{C^{0,\alpha}_{\beta-2;t}} &\leq ct^{-\alpha+\beta}\,||\chi_t \nu - \overline{\nu}||_{C^{0,\alpha}_{0;t}} \cdot \left\|p_L^*\Delta\widetilde{\pi}_t a\right\|_{C^{0,\alpha}_{0;t}} \cdot ||1||_{C^{0,\alpha}_{\beta-2;t}} \\
&\leq ct^{-\alpha+\beta} \cdot t^2 \cdot ||\Delta\widetilde{\pi}_t a||_{C^{0,\alpha}} \\
&\leq ct^{2-\alpha+\beta} \cdot ||\widetilde{\pi}_t a||_{C^{2,\alpha}} \\
&\leq ct^{2-2\alpha+2\beta} \cdot ||a||_{C^{2,\alpha}_{\beta;t}}
\end{aligned}$$

where we used Eqs. (3.59) and (3.60) in the second step; we used $\left\|p_L^*\Delta\widetilde{\pi}_t a\right\|_{C^{0,\alpha}_{0;t}} = ||\Delta\widetilde{\pi}_t a||_{C^{0,\alpha}}$ which holds because $p_L^*\Delta\widetilde{\pi}_t a$ is constant in the Eguchi-Hanson direction, so the derivative in the $C^{0,\alpha}_{0;t}$-norm is just a derivative in the $L$-direction; in the last step we used Eq. (3.80). For the second summand we have

$$t^{2-\alpha+\beta}\,||\widetilde{\pi}_t a||_{C^{2,\alpha}} \leq t^{2-2\alpha+2\beta}\,||a||_{C^{2,\alpha}_{\beta;t}}$$

by Eq. (3.80), which proves the claim. □

**Proof of Proposition 3.52**

*Proof of Proposition 3.52.* By definition, $||a||_{\mathfrak{X}_t} = ||\rho_t a||_{C^{2,\alpha}_{\beta;t}} + t^{-3/2}\,||\pi_t a||_{C^{2,\alpha}}$. We treat the first summand first:

$$\begin{aligned}||\rho_t a||_{C^{2,\alpha}_{\beta;t}} &\leq ||\Delta\rho_t a||_{C^{0,\alpha}_{\beta-2;t}} \\ &\leq \left(||\overline{\pi}_t \Delta\rho_t a||_{C^{0,\alpha}_{\beta-2;t}} + ||\rho_t \Delta a||_{C^{0,\alpha}_{\beta-2;t}} + ||\rho_t \Delta\overline{\pi}_t a||_{C^{0,\alpha}_{\beta-2;t}}\right),\end{aligned}$$

where we used Proposition 3.65 in the first step and in the second step used $1 = \overline{\pi}_t + \rho_t$ twice. Here, the first summand satisfies

$$\begin{aligned}||\overline{\pi}_t \Delta\rho_t a||_{C^{0,\alpha}_{\beta-2;t}} &\leq t^{-\beta}\, ||\pi_t \Delta\rho_t a||_{C^{0,\alpha}} \\ &\leq t^{\beta+2-2\alpha}\, ||\rho_t a||_{C^{2,\alpha}_{\beta;t}},\end{aligned}$$

where we used Proposition 3.48 in the first step, and Eq. (3.79) in the second step. The resulting term can be absorbed into the left hand side of Eq. (3.53).

For the third summand we get from Eq. (3.78) that

$$||\rho_t \Delta\overline{\pi}_t a||_{C^{0,\alpha}_{\beta-2;t}} \leq c t^{2-\alpha}\, ||\pi_t a||_{C^{2,\alpha}},$$

which can be absorbed into the left hand side of Eq. (3.53) if $\alpha$ is sufficiently small. Regarding the $\pi_t$-term, we find that

$$\begin{aligned}t^{-3/2}\, ||\pi_t a||_{C^{2,\alpha}} &\leq t^{-3/2}\, ||\pi_t \Delta\iota_t \pi_t a||_{C^{0,\alpha}} \\ &\leq t^{-3/2}\left(||\pi_t \Delta a||_{C^{0,\alpha}} + ||\pi_t \Delta\rho_t a||_{C^{0,\alpha}}\right),\end{aligned}$$

where we used Proposition 3.61 in the first step and $1 = \overline{\pi}_t + \rho_t$ in the second step. Here we have for the last summand

$$t^{-3/2}\, ||\pi_t \Delta\rho_t a||_{C^{0,\alpha}} \leq t^{-3/2} t^{2+2\beta-2\alpha}\, ||\rho_t a||_{C^{2,\alpha}_{\beta;t}} \tag{3.81}$$

which can be absorbed into the left hand side of Eq. (3.53). The remaining terms, i.e. the ones that have not been absorbed into the left hand side of Eq. (3.53), exactly sum up to $||\Delta a||_{\mathfrak{Y}_t}$, which proves the claim. □

### 3.2.4 The Existence Theorem

We will now prove the theorem which guarantees the existence of a torsion-free $G_2$-structure when starting from a $G_2$-structure with small torsion.

**Theorem 3.82.** *Assume there exists $c > 0$ such that $\psi^t \in \Omega^3(N_t)$ satisfies $\mathrm{d}^*\varphi^t = \mathrm{d}^*\psi^t$ and*

$$\begin{aligned}\left\|\mathrm{d}^*\psi^t\right\|_{\mathfrak{Y}_t} &\leq ct^4,\\ \left\|\psi^t\right\|_{C^{0,\alpha}_{0;t}} &\leq ct^4.\end{aligned}$$

*Then, for small $t$, there exists $\eta^t \in \Omega^2(N_t)$ such that $\varphi^t + \mathrm{d}\eta$ is a torsion-free $G_2$-structure and $\left\|\eta^t\right\|_{\mathfrak{X}_t} \leq ct^4$.*

To ease notation, we write $\varphi = \varphi^t$, $\psi = \psi^t$, and $\eta = \eta^t$ throughout the proof.

*Proof.* We will construct $\eta \in \Omega^2(N_t)$ satisfying

$$\Delta\eta = \mathrm{d}^*\psi + \mathrm{d}^*(f\psi) + *\,\mathrm{d}(F(\mathrm{d}\eta)), \text{ where } f = \frac{7}{3}\langle\varphi, \mathrm{d}\eta\rangle. \tag{3.83}$$

Set $\eta_0 = 0$ and, if $\eta_{j-1} \in \Omega^2(N_t)$ is given, let $\eta_j \in \Omega^2(N_t)$ be such that

$$\Delta\eta_j = \mathrm{d}^*\psi + \mathrm{d}^*(f_{j-1}\psi) + *\,\mathrm{d}\big(F(\mathrm{d}\eta_{j-1})\big), \text{ where } f_{j-1} = \frac{7}{3}\langle\varphi, \mathrm{d}\eta_{j-1}\rangle,$$

and such that $\eta_j \perp \mathcal{K}$. This is well-defined, i.e. such $\eta_j$ exists, because $\operatorname{Im}\mathrm{d}^* \subset \operatorname{Im}\Delta$ and restricting $\Delta$ to $\mathcal{K}^\perp$ does not change its image by Proposition 3.54. We aim to show by induction that $\left\|\eta_j\right\|_{\mathfrak{X}_t} \leq ct^4$. For $j = 0$ this is true by definition, and we will now derive the estimate for $j > 0$.

By definition of $\eta_j$ together with Proposition 3.52 we have that

$$\begin{aligned}\left\|\eta_j\right\|_{\mathfrak{X}_t} &\leq c\left\|\Delta\eta_j\right\|_{\mathfrak{Y}_t}\\ &\leq c\left(||\mathrm{d}^*\psi||_{\mathfrak{Y}_t} + \left\|\mathrm{d}^*(f_{j-1}\psi)\right\|_{\mathfrak{Y}_t} + \left\|*\,\mathrm{d}\big(F(\mathrm{d}\eta_{j-1})\big)\right\|_{\mathfrak{Y}_t}\right)\\ &= c\,(I + II + III)\,.\end{aligned}$$

By assumption we have $I = ||\mathrm{d}^*\psi||_{\mathfrak{Y}_t} \leq ct^4$.

Now to estimate II:

$$\left\|\mathrm{d}^*(f_{j-1}\psi)\right\|_{\mathfrak{Y}_t} \leq \left\|\mathrm{d}f_{j-1} \lrcorner \psi\right\|_{\mathfrak{Y}_t} + \left\|f_{j-1}\,\mathrm{d}^*\psi\right\|_{\mathfrak{Y}_t} = II.A + II.B.$$

Here

$$\begin{aligned} II.A &= \left\|\rho_t(\mathrm{d}f_{j-1} \lrcorner \psi)\right\|_{C^{0,\alpha}_{\beta-2;t}} + t^{-3/2}\left\|\pi_t(\mathrm{d}f_{j-1} \lrcorner \psi)\right\|_{C^{0,\alpha}} \\ &\leq \left(t^{-\alpha} + t^{-3/2-\alpha+\beta}\right)\left\|\mathrm{d}f_{j-1} \lrcorner \psi\right\|_{C^{0,\alpha}_{\beta-2;t}} \\ &\leq \left(t^{-\alpha} + t^{-3/2-\alpha+\beta}\right)\left\|\mathrm{d}f_{j-1}\right\|_{C^{0,\alpha}_{\beta-2;t}} ||\psi||_{C^{0,\alpha}_{0;t}} \\ &\leq ct^4, \end{aligned}$$

where for the first estimate we used Propositions 3.45 and 3.48, and for the last estimate we used the induction hypothesis $\left\|\eta_{j-1}\right\|_{\mathfrak{X}_t} \leq ct^4$, which implies $\left\|\mathrm{d}f_{j-1}\right\|_{C^{0,\alpha}_{\beta-2;t}} \leq ct^{7/2}$, together with the assumption $||\psi||_{C^{0,\alpha}_{0,0;t}} \leq ct^4$. The estimate $II.B \leq ct^4$ is derived analogously.

It remains to estimate III:

$$III = \left\|\rho_t(*\,\mathrm{d}\big(F(\mathrm{d}\eta_{j-1})\big)\right\|_{C^{0,\alpha}_{\beta-2;t}} + t^{-3/2}\left\|\pi_t(*\,\mathrm{d}\big(F(\mathrm{d}\eta_{j-1})\big)\right\|_{C^{0,\alpha}} = III.A + III.B.$$

The summand III.A is estimated as

$$III.A \leq ct^{-\alpha}\left\|*\,\mathrm{d}\big(F(\mathrm{d}\eta_{j-1})\big)\right\|_{C^{0,\alpha}_{\beta-2;t}},$$

where we first estimate the $L^\infty$-part of the $C^{0,\alpha}$-norm. Namely, by Proposition 2.24:

$$\begin{aligned} \left\|*\,\mathrm{d}\big(F(\mathrm{d}\eta_{j-1})\big)\right\|_{L^\infty_{\beta-2;t}} &\leq c\left\|\mathrm{d}\eta_{j-1}\right\|_{L^\infty_{\beta-1;t}}\left\|\nabla\,\mathrm{d}\eta_{j-1}\right\|_{L^\infty_{\beta-2;t}} t^{-1+\beta} \\ &\quad + c\left\|\mathrm{d}\eta_{j-1}\right\|^2_{L^\infty_{\beta-1;t}} ||\mathrm{d}^*\psi||_{L^\infty_{\beta-2;t}} t^{-2+2\beta} \\ &\leq ct^4. \end{aligned}$$

The $[\cdot]_{C^{0,\alpha}}$-part is estimated analogously. To estimate $III.B = t^{-3/2}\left\|\pi_t\left(*\,\mathrm{d}\big(F(\mathrm{d}\eta_{j-1})\big)\right)\right\|_{C^{0,\alpha}}$, we again estimate the $L^\infty$-part first. Fix some $y \in L$ and compute $\pi_t\left(*\,\mathrm{d}\big(F(\mathrm{d}\eta_{j-1})\big)\right)(y)$ by computing an integral over $X_{\mathrm{EH}} \simeq \{y\} \times X_{\mathrm{EH}} \subset L \times X_{\mathrm{EH}}$. By Proposition 2.24 we have

$$\begin{aligned}
\left|\pi_t\left(*\,\mathrm{d}\big(F(\mathrm{d}\eta_{j-1})\big)\right)\right| &\leq \left|\left\langle *\,\mathrm{d}\big(F(\mathrm{d}\eta_{j-1})\big), \chi_t \nu\right\rangle_{t^2 g_{X_{\mathrm{EH}}}}\right| \\
&\leq \underbrace{c\int_{X_{\mathrm{EH}}} |\,\mathrm{d}\eta_{j-1}| \cdot |\nabla\,\mathrm{d}\eta_{j-1}| \cdot |\chi_t \nu| \operatorname{vol}_{t^2 g_{X_{\mathrm{EH}}}}}_{III.B.1} \\
&\quad + \underbrace{c\int_{X_{\mathrm{EH}}} |\,\mathrm{d}\eta_{j-1}| \cdot |\,\mathrm{d}\eta_{j-1}| \cdot |\,\mathrm{d}^*\psi| \cdot |\chi_t \nu| \operatorname{vol}_{t^2 g_{X_{\mathrm{EH}}}}}_{III.B.2}.
\end{aligned}$$

Here,

$$\begin{aligned}
III.B.1 \cdot t^{3/2} &= c\int_{X_{\mathrm{EH}}} |\,\mathrm{d}(\overline{\pi}_t\eta_{j-1} + \rho_t\eta_{j-1})| \cdot |\nabla\,\mathrm{d}(\overline{\pi}_t\eta_{j-1} + \rho_t\eta_{j-1})| \cdot |\chi_t\nu| \operatorname{vol}_{t^2 g_{X_{\mathrm{EH}}}} \\
&\leq c\int_0^\zeta \Big((t+r)^{-7} \underbrace{\left\|\mathrm{d}\overline{\pi}_t\eta_{j-1}\right\|_{C^{0,\alpha}_{-3;t}} \left\|\nabla\,\mathrm{d}\overline{\pi}_t\eta_{j-1}\right\|_{C^{0,\alpha}_{-4;t}}}_{\leq c\left\|\pi_t\eta_{j-1}\right\|^2_{C^{2,\alpha}} \leq ct^{2\cdot(4+3/2)}}\Big)\left((t+r)^{-4}t^2\right) r^3\,\mathrm{d}r \\
&\quad + c\int_0^\zeta \Big((t+r)^{2\beta-3} \underbrace{\left\|\mathrm{d}\rho_t\eta_{j-1}\right\|_{C^{0,\alpha}_{\beta-1;t}} \left\|\nabla\,\mathrm{d}\rho_t\eta_{j-1}\right\|_{C^{0,\alpha}_{\beta-2;t}}}_{\leq c\left\|\rho_t\eta_{j-1}\right\|^2_{C^{2,\alpha}_{\beta;t}} \leq ct^{2\cdot 4}}\Big)\Big((t+r)^{-4}t^2\Big) r^3\,\mathrm{d}r \\
&\quad + c\int_0^\zeta \Big((t+r)^{\beta-5} \underbrace{\left\|\mathrm{d}\overline{\pi}_t\eta_{j-1}\right\|_{C^{0,\alpha}_{-3;t}} \left\|\nabla\,\mathrm{d}\rho_t\eta_{j-1}\right\|_{C^{0,\alpha}_{\beta-2;t}}}_{\leq c\left\|\pi_t\eta_{j-1}\right\|_{C^{2,\alpha}}\left\|\rho_t\eta_{j-1}\right\|_{C^{2,\alpha}_{\beta;t}} \leq ct^{4+3/2+4}}\Big)\left((t+r)^{-4}t^2\right) r^3\,\mathrm{d}r \\
&\quad + c\int_0^\zeta \Big((t+r)^{\beta-5} \underbrace{\left\|\mathrm{d}\rho_t\eta_{j-1}\right\|_{C^{0,\alpha}_{\beta-1;t}} \left\|\nabla\,\mathrm{d}\overline{\pi}_t\eta_{j-1}\right\|_{C^{0,\alpha}_{-4;t}}}_{\leq c\left\|\pi_t\eta_{j-1}\right\|_{C^{2,\alpha}}\left\|\rho_t\eta_{j-1}\right\|_{C^{2,\alpha}_{\beta;t}} \leq ct^{4+3/2+4}}\Big)\left((t+r)^{-4}t^2\right) r^3\,\mathrm{d}r \\
&\leq c\left(t^{2\cdot(4+3/2)}t^{-7}t^2 + t^{2\cdot 4}t^{2\beta-3}t^2 + 2t^{4+3/2+4}t^{\beta-5}t^2\right) \\
&\leq ct^6,
\end{aligned}$$

thus $III.B.1 \leq ct^4$. The part $III.B.2$ and the $C^{0,\alpha}$-parts of $III.B.1$ and $III.B.2$ are estimated analogously. Altogether, this gives $III \leq ct^4$.

The sequence $\eta_j$ satisfies

$$\begin{aligned}\|\eta_j\|_{C^{2,\alpha}_{\beta;t}} &\le \|\rho_t\eta_j\|_{C^{2,\alpha}_{\beta;t}} + \|\overline{\pi}_t\eta_j\|_{C^{2,\alpha}_{\beta;t}} \\ &\le \|\eta_j\|_{\mathfrak{X}_t} + t^{-2-\beta+3/2}\|\eta_j\|_{\mathfrak{X}_t} \\ &\le ct^{7/2-\beta}.\end{aligned}$$

As usual, the constant $c$ is independent of $t$, but in particular independent of $j$. Thus, there exists, up to a subsequence, a $C^{2,\alpha/2}$-limit $\lim_{j\to\infty}\eta_j =: \eta$ by the Arzelà–Ascoli theorem. This limit solves Eq. (3.83) and satisfies

$$\|\eta\|_{C^{2,\alpha/2}_{\beta;t}} \le ct^{7/2-\beta}.$$

By [Joy00][Theorem 10.3.7], $\varphi + \mathrm{d}\eta$ is a torsion-free $G_2$-structure, which proves the claim. □

Taking everything together, this gives us:

**Theorem 3.84.** *Let $N_t$ be the resolution of $T^7/\Gamma$ from Eq.* (3.31) *and $\varphi^t \in \Omega^3(N_t)$ the $G_2$-structure with small torsion from Eq.* (3.33). *There exists $c > 0$ independent of $t$ such that the following is true: for $t$ small enough, there exists $\eta^t \in \Omega^2(N_t)$ such that $\widetilde{\varphi} = \varphi^t + \mathrm{d}\eta^t$ is a torsion-free $G_2$-structure, and $\eta^t$ satisfies*

$$\|\eta^t\|_{C^{2,\alpha/2}_{\beta;t}} \le ct^{7/2-\beta}.$$

*In particular,*

$$\|\widetilde{\varphi} - \varphi^t\|_{L^\infty} \le ct^{5/2} \text{ and } \|\widetilde{\varphi} - \varphi^t\|_{C^{0,\alpha/2}} \le ct^{5/2-\alpha/2} \text{ as well as } \|\widetilde{\varphi} - \varphi^t\|_{C^{1,\alpha/2}} \le ct^{3/2-\alpha/2}.$$

*Proof.* By Lemma 3.36, we have that $\|\psi\|_{C^{0,\alpha}_{0;t}} \le ct^4$. Combined with Propositions 3.45 and 3.48, we also have $\|\psi\|_{\mathfrak{Y}_t} \le ct^4$. Thus, Theorem 3.82 can be applied, which gives the claim. □

*Remark* 3.85. The power $7/2 - \beta$ in Theorem 3.84 can be improved to $4 - \epsilon$ for any $\epsilon \in (0,1)$ by defining the norms $\|\cdot\|_{\mathfrak{X}_t}$ and $\|\cdot\|_{\mathfrak{Y}_t}$ with a factor of $t^{-\kappa}$ instead of $t^{-3/2}$ for $\kappa \in (0,2)$ close to 2.

*Remark* 3.86. In [Joy96a], compact manifolds with holonomy Spin(7) were constructed. In the simplest case, one constructs Spin(7)-structures with small torsion by gluing together the product Spin(7)-structure on $T^4 \times X_{\mathrm{EH}}$ and the flat Spin(7)-structure on $T^8$. This gluing construction is analogue to the definition of the $G_2$-structure in Eq. (3.33). In contrast to the $G_2$-situation, however, Joyce's theorem about the existence of torsion-free Spin(7)-structures cannot immediately be applied, because the torsion of the glued structure is too big. He overcame this problem by constructing a correction of the glued structure by hand which has smaller torsion, to which the existence theorem can be applied. The same can be done in the $G_2$ case. In fact, one gets a correction in the $G_2$-case from the Spin(7)-case by considering the Spin(7)-orbifold $T^7/\Gamma \times S^1$. Using this corrected structure, one would get even better control over the difference between glued structure and torsion-free structure than what is known from Theorem 3.84.

## 3.3 Torsion-Free $G_2$-Structures on Joyce-Karigiannis Manifolds

In [JK21], the authors constructed new examples of compact manifolds with holonomy $G_2$ by generalising the construction that was described in Section 3.2.1. As in Section 3.2, they first use a gluing procedure to construct a $G_2$-structure with small torsion. They then apply Theorem 2.26 to perturb this $G_2$-structure into a torsion-free $G_2$-structure.

The main difference to Joyce's original construction is the following: if one uses the cutoff procedure from the $T^7/\Gamma$ case in the new setting, one produces a $G_2$-structure that does not satisfy the necessary estimates to apply Theorem 2.26. The authors of [JK21] overcome this problem by constructing a $G_2$-structure with *even* smaller torsion, to which Theorem 2.26 *can* be applied.

### 3.3.1 Ingredients for the Construction

Let $Y$ be a compact manifold endowed with a torsion-free $G_2$-structure $\varphi$. Write $g$ for the metric induced by $\varphi$. Let $\iota : Y \to Y$ be a $G_2$-involution, i.e. satisfying $\iota^2 = \mathrm{Id}$, $\iota \neq \mathrm{Id}$, $\iota^*\varphi = \varphi$. We then have:

**Proposition 3.87** (Proposition 2.13 in [JK21]). *Let $L = \mathrm{fix}(\iota)$ and assume $L \neq \emptyset$. Then $L$ is a* *smooth, orientable* 3*-dimensional compact submanifold of* $Y$ *which is totally geodesic, and, with respect to a canonical orientation, is associative.*

*Assumption* 3.88. We assume that $L$ is nonempty, and we assume we are given a closed, coclosed, nowhere vanishing 1-form $\lambda$ on $L$.

Such a 1-form need not exist, and cases in which its existence can be guaranteed are discussed in [JK21, Section 7.1].

### 3.3.2 $G_2$-structures on the Normal Bundle $\nu$ of $L$

The metric defined by $\varphi$ defines a splitting

$$TY|_L \simeq \nu \oplus TL, \tag{3.89}$$

which is orthogonal with respect to $g$. Write $g_L$ for the metric on $L$ induced by $g$ and $g|_L = h_\nu \oplus g_L$. Write $\widetilde{\nabla}^\nu$ for some connection on $\nu$. For now, we may think of $\widetilde{\nabla}^\nu$ as being the restriction of the Levi-Civita connection of $g$ to $\nu \to L$, but later we will need the freedom to choose another connection. We write elements in $\nu$ as $(x, \alpha)$, where $x \in L$, $\alpha \in \nu_x$. For $R > 0$ let

$$U_R = \{(x, \alpha) \in \nu : |\alpha|_{h_\nu} < R\}.$$

Write $\pi : U_R \to L$ for the projection $(x, \alpha) \mapsto x$. We will make use of a map $\Upsilon : U_R \to Y$ satisfying the following:

1. $\Upsilon$ is a diffeomorphism onto its image,
2. $\Upsilon(x, 0) = x$ for $x \in L$,
3. $\Upsilon(x, -\alpha) = \iota \circ \Upsilon(x, \alpha)$ for $(x, \alpha) \in U_R$,
4. the induced pushforward $\Upsilon_* : TU_R \to TY$ restricted to the zero section of $TU_R$ is the identity map on $T_xL \oplus \nu_x$.

For example, $\Upsilon = \exp$ would satisfy these four conditions for small $R$. But later on we require $\Upsilon$ to satisfy an extra condition that exp need not satisfy.

Write $(\cdot t) : \nu \to \nu$ for the dilation map $(x, \alpha) \mapsto (x, t\alpha)$, and for $t \neq 0$, define $\Upsilon_t = \Upsilon \circ (\cdot t) : U_{|t|^{-1}R} \to Y$.

The connection $\widetilde{\nabla}^\nu$ defines a splitting

$$T\nu = V \oplus H, \quad \text{where } V \simeq \pi^*(\nu) \text{ and } H \simeq \pi^*(TL), \tag{3.90}$$

where $V$ and $H$ are the vertical and horizontal subbundles of the connection. Combining Eqs. (3.89) and (3.90), we have that $T\nu \simeq \pi^*(TY|_L)$. Denote by

$$\varphi^\nu \in \Omega^3(\nu), \psi^\nu \in \Omega^4(\nu), \text{ and } g^\nu \in S^2(\nu) \tag{3.91}$$

the structures obtained from $\varphi$, $\psi$, and $g$ via this isomorphism and for $t > 0$ write $\varphi_t^\nu = (\cdot t)^*\varphi^\nu$, as well as $\psi_t^\nu = (\cdot t)^*\psi^\nu$, and $g_t^\nu = (\cdot t)^* g^\nu$. Note that this definition implicitly depends on the choice of $\widetilde{\nabla}^\nu$. The main result of [JK21, Section 3] is then:

**Proposition 3.92.** *There exist $R > 0$, a connection $\widetilde{\nabla}^\nu$ on $\nu$ and a map $\Upsilon : U_R \to M$ satisfying*

1. *$\Upsilon$ is a diffeomorphism onto its image,*
2. *$\Upsilon(x, 0) = x$ for $x \in L$,*
3. *$\Upsilon(x, -\alpha) = \iota \circ \Upsilon(x, \alpha)$ for $(x, \alpha) \in U_R$,*
4. *the induced pushforward $\Upsilon_* : TU_R \to TY$ restricted to the zero section of $TU_R$ is the identity map on $T_xL \oplus \nu_x$,*

*and for $t > 0$ a closed $G_2$-structure $\widetilde{\varphi}_t^\nu$ on $\nu/\{\pm 1\}$ and closed 4-form $\widetilde{\psi}_t^\nu \in \Omega^4(\nu/\{\pm 1\})$ satisfying the following properties: first,*

$$\varphi_t^\nu - \widetilde{\varphi}_t^\nu = O(t^2 r^2) \quad \textit{and} \quad \psi_t^\nu - \widetilde{\psi}_t^\nu = O(t^2 r^2). \tag{3.93}$$

*Second, there exist $\eta \in \Omega^2(\nu), \zeta \in \Omega^3(\nu)$ such that*

$$|\eta|_{g^\nu} = O(r^3) \qquad \textit{and} \qquad |\,\mathrm{d}\eta|_{g^\nu} = \left|\Upsilon^*\varphi - \widetilde{\varphi}^\nu|_{U_R}\right|_{g^\nu} = O(r^2),$$
$$|\zeta|_{g^\nu} = O(r^3) \qquad \textit{and} \qquad |\,\mathrm{d}\zeta|_{g^\nu} = \left|\Upsilon^*\psi - \widetilde{\psi}^\nu|_{U_R}\right|_{g^\nu} = O(r^2).$$

### 3.3.3 $G_2$-structures on the Resolution $P$ of $\nu/\{\pm 1\}$

The $G_2$-structure $\varphi \in \Omega^3(Y)$ defines for all $x \in Y$ a cross product $\times : T_xY \times T_xY \longrightarrow T_xY$ as in Definition 2.19. We then have a complex structure $I \in \mathrm{End}(\nu)$ given by

$$I(V) = \frac{\lambda}{|\lambda|} \times V \text{ for } V \in \nu_x, x \in L. \tag{3.94}$$

Recall the metric $h_\nu$ on $\nu$ defined by $g|_L = h_\nu \oplus g_L$, cf. Section 3.3.2. Then $I$ and $h_\nu$ together define a $\mathrm{U}(2)$-reduction of the frame bundle of $\nu$. Denote by $X_{\mathrm{EH}}$ the Eguchi-Hanson space with Hyperkähler triple $\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}$ from Proposition 2.10. Denote by $\rho : X_{\mathrm{EH}} \longrightarrow \mathbb{C}^2/\{\pm 1\}$ the blowup map of the blowup with respect to the complex structure induced by $\omega_1^{(1)}$ from Remark 2.13 and let

$$P = \mathrm{Fr} \times_{\mathrm{U}(2)} X_{\mathrm{EH}}. \tag{3.95}$$

Denote by $\sigma : P \longrightarrow L$ the projection of this bundle. Analogously, we have

$$\nu/\{\pm 1\} = \mathrm{Fr} \times_{\mathrm{U}(2)} \mathbb{C}^2/\{\pm 1\}.$$

Let $L' \subset L$ be a nonempty, open set on which we can extend $e_1 := \frac{\lambda}{|\lambda|} \in T^*(L')$ to an orthonormal basis $(e_1, e_2, e_3)$. Then there exist $\hat{\omega}^I, \hat{\omega}^J, \hat{\omega}^K \in \Omega^2((\nu/\{\pm 1\})|_{L'})$ such that $\varphi^\nu$ from Eq. (3.91) has the form

$$\varphi^\nu = e_1 \wedge e_2 \wedge e_3 - \hat{\omega}^I \wedge e_1 - \hat{\omega}^J \wedge e_2 - \hat{\omega}^K \wedge e_3. \tag{3.96}$$

We define $\check{\omega}^I, \check{\omega}^J, \check{\omega}^K \in \Omega^2(P|_{L'})$ as follows: For $x \in L'$, let $f \in \mathrm{Fr}_x$ such that $f : (\nu/\{\pm 1\})_x \to \mathbb{C}^2/\{\pm 1\}$ satisfies

$$f^*(\omega_1^{(0)}, \omega_2^{(0)}, \omega_3^{(0)}) = (\hat{\omega}^I|_{\nu_x}, \hat{\omega}^J|_{\nu_x}, \hat{\omega}^K|_{\nu_x}),$$

where $(\omega_1^{(0)}, \omega_2^{(0)}, \omega_3^{(0)})$ denotes the Hyperkähler triple on $\mathbb{C}^2/\{\pm 1\}$ from Proposition 2.10. This choice of $f$ defines isomorphisms of complex surfaces $P_x \simeq X_{\mathrm{EH}}$ and $(\nu/\{\pm 1\})_x \simeq \mathbb{C}^2/\{\pm 1\}$. Let $\check{\omega}^I, \check{\omega}^J, \check{\omega}^K \in \Omega^2(P_x)$ be the pullback of $\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)} \in \Omega^2(X_{\mathrm{EH}})$ under this isomorphism. This is independent of the choice of $f$, and therefore defines $\check{\omega}^I, \check{\omega}^J, \check{\omega}^K \in \Omega^2(P_x)$. The following diagram sums up the situation:

$$\begin{tikzcd}
(P_x, \check{\omega}^I|_{P_x}, \check{\omega}^J|_{P_x}, \check{\omega}^K|_{P_x}) \arrow[r, "\simeq"] \arrow[d, "\rho"] & (X_{\mathrm{EH}}, \omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}) \arrow[d, "\rho"] \\
(\nu_x/\{\pm 1\}, \hat{\omega}^I|_{\nu_x/\{\pm 1\}}, \hat{\omega}^J|_{\nu_x/\{\pm 1\}}, \hat{\omega}^K|_{\nu_x/\{\pm 1\}}) \arrow[r, "\simeq"] & (\mathbb{C}^2/\{\pm 1\}, \omega_1^{(0)}, \omega_2^{(0)}, \omega_3^{(0)})
\end{tikzcd} \tag{3.97}$$

Here, by abuse of notation we denoted the map $P_x \to \nu_x/\{\pm 1\}$ which makes the diagram commutative also by $\rho$. Horizontal arrows pull Hyperkähler triples back to one another, Hyperkähler triples connected by vertical arrows are asymptotic in the sense of Proposition 2.10. A complicated point is the actual definition of $\check{\omega}^I, \check{\omega}^J, \check{\omega}^K$ as 2-forms on $P|_{L'}$. Equation (3.97) tells us what they look like fibrewise. To make sense of them as global objects on $P$, one needs to choose a connection on $P$. In [JK21], the horizontal subspaces $\check{H}$ were defined to this end which allows us to decompose forms on $P$ into vertical and horizontal components, much like for forms on $\nu$. There are then unique vertical 2-forms which restrict to $\check{\omega}^I|_{P_x}, \check{\omega}^J|_{P_x}, \check{\omega}^K|_{P_x}$ on every fibre.

We are now ready to define $\varphi_t^P \in \Omega^3(P|_{L'})$, $\psi_t^P \in \Omega^4(P|_{L'})$ via

$$
\begin{aligned}
\varphi_t^P &:= \check{\varphi}_{0,3} + t^2\check{\varphi}_{2,1} \\
&:= \sigma^*(e_1 \wedge e_2 \wedge e_3) - t^2\left(\sigma^*(e_1) \wedge \check{\omega}^I - \sigma^*(e_2) \wedge \check{\omega}^J - \sigma^*(e_3) \wedge \check{\omega}^K\right),
\end{aligned}
\tag{3.98}
$$

$$
\begin{aligned}
\psi_t^P &:= t^4\check{\psi}_{4,0} + t^2\check{\psi}_{2,2} \\
&:= \frac{1}{2}\check{\omega}^I \wedge \check{\omega}^I - \sigma^*(e_2 \wedge e_3) \wedge \check{\omega}^I - \sigma^*(e_3 \wedge e_1) \wedge \check{\omega}^J - \sigma^*(e_1 \wedge e_2) \wedge \check{\omega}^K.
\end{aligned}
$$

These expressions are independent of the choice of $(e_2, e_3)$, and therefore define forms $\varphi_t^P \in \Omega^3(P)$, $\psi_t^P \in \Omega^4(P)$, not just forms over $L' \subset L$. Let also $g_t^P$ denote the metric induced by $\varphi_t^P$.

As in the previous section, we add terms to $\varphi_t^P$ and $\psi_t^P$ to define *closed* forms on $P$, and we have the following control over how they are asymptotic to forms on $\nu/\{\pm 1\}$:

**Proposition 3.99** (Section 4.5 in [JK21]). *There exist $\xi_{1,2}, \xi_{0,3} \in \Omega^3(P)$, $\tau_{1,1} \in \Omega^2(\{x \in P : \check{r}(x) > 1)$, such that*

$$\widetilde{\varphi}_t^P := \varphi_t^P + t^2\xi_{1,2} + t^2\xi_{0,3}$$

*is closed and satisfies*

$$\widetilde{\varphi}_t^P = \rho^*\widetilde{\varphi}_t^\nu + t^2\,\mathrm{d}\tau_{1,1} \tag{3.100}$$

*where $\check{r} > 1$. These forms satisfy the following estimates:*

$$
\begin{aligned}
\left|\nabla^k(t^2\xi_{1,2})\right|_{g_t^P} &= \begin{cases} O(t^{1-k}), & \check{r} \leq 1, \\ O(t^{1-k}\check{r}^{-3-k}), & \check{r} > 1, \end{cases} \\
\left|\nabla^k(t^2\xi_{0,3})\right|_{g_t^P} &= \begin{cases} O(t^{2-k}), & \check{r} \leq 1, \\ O(t^{2-k}\check{r}^{2-k}), & \check{r} > 1, \end{cases}
\end{aligned}
\tag{3.101}
$$

$$\left|\nabla^k(t^2\tau_{1,1})\right|_{g_t^P} = O(t^{1-k}\check{r}^{-3-k}). \tag{3.102}$$

**Proposition 3.103** (Section 4.5 in [JK21]). *There exist $\chi_{1,3}, \theta_{3,1}, \theta_{2,2} \in \Omega^4(P)$, $v_{1,2} \in \Omega^3(\{x \in P :$*

$\check{r}(x) > 1$), *such that*

$$\widetilde{\psi}_t^P := \psi_t^P + t^2\chi_{1,3} + t^4\theta_{3,1} + t^4\theta_{2,2} \tag{3.104}$$

*is closed and satisfies*

$$\widetilde{\psi}_t^P = \rho^*\widetilde{\psi}_t^\nu + t^2\,\mathrm{d}v_{1,2} \tag{3.105}$$

*where* $\check{r} > 1$. *These forms satisfy the following estimates:*

$$\left|\nabla^k(t^2\chi_{1,3})\right|_{g_t^P} := \begin{cases} O(t^{1-k}), & \check{r} \leq 1, \\ O(t^{1-k}\check{r}^{-3-k}), & \check{r} > 1, \end{cases} \tag{3.106}$$

$$\left|\nabla^k(t^4\theta_{3,1})\right|_{g_t^P} := \begin{cases} O(t^{1-k}), & \check{r} \leq 1, \\ 0, & \check{r} > 1, \end{cases} \tag{3.107}$$

$$\left|\nabla^k(t^4\theta_{2,2})\right|_{g_t^P} := \begin{cases} O(t^{2-k}), & \check{r} \leq 1, \\ O(t^{2-k}\check{r}^{2-k}), & \check{r} > 1, \end{cases} \tag{3.108}$$

$$\left|\nabla^k(t^2 v_{1,2})\right|_{g_t^P} := O(t^{1-k}\check{r}^{-3-k}). \tag{3.109}$$

### 3.3.4 Correcting for the Leading-order Errors on $P$

Armed with the $G_2$-structures $\varphi$ on $Y$ and $\widetilde{\varphi}_t^P$ on $P$, we could define a glued together $G_2$-structure just as we did in Section 3.2. However, in this case it would turn out that the torsion of the glued together $G_2$-structure is too big and Theorem 2.26 cannot be applied. We thus make use of the following correction terms which will make the torsion of the glued together $G_2$-structure small enough.

**Theorem 3.110** (Theorem 5.1 in [JK21]). *There exist* $\alpha_{0,2}, \alpha_{2,0} \in \Omega^2(P)$, $\beta_{0,3}, \beta_{2,1} \in \Omega^3(P)$, *satisfying for all* $t > 0$ *the equation*

$$(D_{\varphi_t^P}\Theta)\left(t^2[\mathrm{d}\alpha_{0,2}]_{1,2} + t^4[\mathrm{d}\alpha_{2,0}]_{3,0} + t^2\xi_{1,2}\right) = t^2\,\mathrm{d}\beta_{0,3} + t^4[\mathrm{d}\beta_{2,1}]_{3,1} + t^2\chi_{1,3} + t^4\theta_{3,1}.$$

*Moreover, for $\gamma > 0$ sufficiently small and for all $k \geq 0$, these forms satisfy the following estimates*

$$\left|\nabla^k(t^2\alpha_{0,2})\right|_{g_t^P} = \begin{cases} O(t^{2-k}), & \check{r} \leq 1, \\ O(t^{2-k}\check{r}^{-2-k+\gamma}), & \check{r} \geq 1, \end{cases}$$

$$\left|\nabla^k(t^4\alpha_{2,0})\right|_{g_t^P} = \begin{cases} O(t^{2-k}), & \check{r} \leq 1, \\ O(t^{2-k}\check{r}^{-2-k+\gamma}), & \check{r} \geq 1, \end{cases}$$

$$\left|\nabla^k(t^2\beta 0, 3)\right|_{g_t^P} = \begin{cases} O(t^{2-k}), & \check{r} \leq 1, \\ O(t^{2-k}\check{r}^{-2-k+\gamma}), & \check{r} \geq 1, \end{cases}$$

$$\left|\nabla^k(t^4\beta 2, 1)\right|_{g_t^P} = \begin{cases} O(t^{2-k}), & \check{r} \leq 1, \\ O(t^{2-k}\check{r}^{-2-k+\gamma}), & \check{r} \geq 1, \end{cases}$$

### 3.3.5 $G_2$-structures on the Resolution $N_t$ of $Y/\langle\iota\rangle$

We are now ready to glue together $P$ and $Y/\langle\iota\rangle$ to a manifold, and define a $G_2$-structure with small torsion on it.

*Definition* 3.111. Define

$$N_t := \left[\rho^{-1}(U_{t^{-1}R}/\{\pm 1\}) \coprod (Y \setminus L)/\langle\iota\rangle\right]/\sim, \tag{3.112}$$

where $x \sim \Upsilon_t \circ \rho(x)$ for $x \in \rho^{-1}(U_{t^{-1}R}/\{\pm 1\})$.

*Definition* 3.113. Let $a : [0, \infty) \to \mathbb{R}$ be a smooth function with $a(x) = 0$ for $x \in [0, 1]$, and

$a(x) = 1 \in [2, \infty)$. Define then

$$\varphi_t^N = \begin{cases} \widetilde{\varphi}_t^P + \mathrm{d}[t^2\alpha_{0,2} + t^4\alpha_{2,0}], & \text{if } \check{r} \le t^{-1/9}, \\ \widetilde{\varphi}_t^P + \mathrm{d}[t^2\alpha_{0,2} + t^4\alpha_{2,0} + a(t^{1/9}\check{r}) \cdot \Upsilon_*\eta], & \text{if } t^{-1/9} \le \check{r} \le 2t^{-1/9}, \\ \widetilde{\varphi}_t^P + \mathrm{d}[t^2\alpha_{0,2} + t^4\alpha_{2,0} + \Upsilon_*\eta], & \text{if } 2t^{-1/9} \le \check{r} \le t^{-4/5}, \\ \widetilde{\varphi}_t^\nu + \mathrm{d}[(1 - a(t^{4/5}\check{r}))(t^2\tau_{1,1} + t^2\alpha_{0,2} + t^4\alpha_{2,0}) + \Upsilon_*\eta], & \text{if } t^{-4/5} \le \check{r} \le 2t^{-4/5}, \\ \varphi, & \text{elsewhere,} \end{cases} \tag{3.114}$$

$$\psi_t^N = \begin{cases} \widetilde{\psi}_t^P + \mathrm{d}[t^2\beta_{0,3} + t^4\beta_{2,1}], & \text{if } \check{r} \le t^{-1/9}, \\ \widetilde{\psi}_t^P + \mathrm{d}[t^2\beta_{0,3} + t^4\beta_{2,1} + a(t^{1/9}\check{r}) \cdot \Upsilon_*\zeta], & \text{if } t^{-1/9} \le \check{r} \le 2t^{-1/9}, \\ \widetilde{\psi}_t^P + \mathrm{d}[t^2\beta_{0,3} + t^4\beta_{2,1} + \Upsilon_*\zeta], & \text{if } 2t^{-1/9} \le \check{r} \le t^{-4/5}, \\ \widetilde{\psi}_t^\nu + \mathrm{d}[(1 - a(t^{4/5}\check{r}))(t^2 v_{1,2} + t^2\beta_{0,3} + t^4\beta_{2,1}) + \Upsilon_*\zeta], & \text{if } t^{-4/5} \le \check{r} \le 2t^{-4/5}, \\ \psi, & \text{elsewhere,} \end{cases} \tag{3.115}$$

The important properties of these forms are that $\varphi_t^N$ and $\psi_t^N$ are closed, and that $\psi_t^N$ is close to being the Hodge dual of $\varphi_t^N$. That is, the 3-form $\varphi_t^N - *_{\varphi_t^N}\psi_t^N$ satisfies the assumption of Theorem 2.26 and $\varphi_t^N$ can be perturbed to a torsion-free $G_2$-structure. This yields the following theorem:

**Theorem 3.116** (Theorem 6.4 in [JK21]). *For small $t$ there exists $\eta_t \in \Omega^2(N_t)$ such that $\widetilde{\varphi}_t^N := \varphi_t^N + \mathrm{d}\eta_t$ is a torsion-free $G_2$-structure, and*

$$\left\|\widetilde{\varphi}_t^N - \varphi_t^N\right\|_{L^\infty} \le ct^{1/18} \tag{3.117}$$

*for some constant $c > 0$ independent of $t$.*

# 4 The Gluing Construction for Instantons

We now turn to constructing $G_2$-instantons on the resolutions of $Y/\langle\iota\rangle$ explained in the previous chapter. Much like explained in the introduction to Section 3, we will follow again the three step process of (1) constructing an approximate solution, (2) estimating the linearisation of the equation to be solved, and (3) perturbing the approximate solution to a genuine solution.

In Section 4.1 we explain how a section $s$ of a moduli bundle gives rise to a connection $s(A)$ on the bundle of Eguchi-Hanson spaces $P$ from Eq. (3.95), cf. Theorem 4.15. If the topological compatibility condition Assumption 4.1 is satisfied, we can glue $s(A)$ to a $G_2$-instanton $\theta$ on the orbifold $Y/\langle\iota\rangle$. The resulting connection $A_t$ is close to being a $G_2$-instanton and in Section 4.2 we will quantify this. We will see that this error is small in a suitable norm if $s$ satisfies a first order partial differential equation, the Fueter equation. Section 4.3 is the difficult part of the analysis, where we give an estimate for the inverse of the linearised instanton operator. In Sections 4.4 and 4.5 we complete the argument and construct the perturbation that turns the approximate solution from before into a genuine solution to the $G_2$-instanton equation.

Throughout we will use the notation from the previous chapter. That is, $Y$ is a $G_2$-manifold with $G_2$-involution $\iota : Y \to Y$, and $N_t$ is the resolution of $Y/\langle\iota\rangle$. The resolution $N_t$ is obtained by gluing in the Eguchi-Hanson bundle $P$ over the singular locus $L = \operatorname{fix}(\iota)$. On $P$ we have the $G_2$-structures $\varphi_t^P$ and $\widetilde{\varphi}_t^P$, and on $N_t$ we have the $G_2$-structure $\varphi_t^N$ with small torsion and the torsion-free $G_2$-structure $\widetilde{\varphi}_t^N$. In the case that $N_t$ is a resolution of $T^7/\Gamma$, we also defined the $G_2$-structures $\varphi^t$ and $\widetilde{\varphi}^t$. These two will also be denoted by $\varphi_t^N$ and $\widetilde{\varphi}_t^N$ respectively and the special case of $T^7/\Gamma$ will need no special treatment most of the time. The exception is the pre-gluing estimate for resolutions of $T^7/\Gamma$, Corollary 4.57, which is better than in the general case. In the case of resolutions of $T^7/\Gamma$, our main result is Theorem 4.131:

**Theorem.** *Let $N \to Y'$ be the resolution of the orbifold $Y' = T^7/\Gamma$ from before. Assume that the connection $\theta$ used to define the approximate $G_2$-instanton $A_t$ from Proposition 4.27 is infinitesimally rigid and that $s$ is an infinitesimally rigid Fueter section.*

*There exists $c > 0$ such that for small $t$ there exists an $\underline{a}_t = (a_t, \xi_t) \in C^{1,\alpha}(\Omega^0 \oplus \Omega^1(\operatorname{Ad} E_t))$ such that $\widetilde{A}_t := A_t + a_t$ is a $G_2$-instanton. Furthermore, $\underline{a}_t$ satisfies $\|\underline{a}_t\|_{\mathfrak{X}_t} \leq ct^{2-2\alpha}$.*

Here, $||\cdot||_{\mathfrak{X}_t}$ is a complicated composite norm similar to the norm denoted with the same symbol from Section 3, and $\alpha \in (0,1)$ is a number close to 0. In the general case of resolutions of $Y/\langle\iota\rangle$ we only have a weaker result. Namely, we require the Fueter section to be pointwise rigid. This is Theorem 4.130:

**Theorem.** *Assume now that the section $s$ is given by a rigid ASD-instanton in every point $x \in L$, and assume that the connection $\theta$ used to define the approximate $G_2$-instanton $A_t$ from Proposition 4.27 is infinitesimally rigid.*

*There exists $c > 0$ such that for small $t$ there exists $\underline{a}_t = (a_t, \xi_t) \in C^{1,\alpha}(\Omega^0 \oplus \Omega^1(\mathrm{Ad}\, E_t))$ such that $\widetilde{A}_t := A_t + a_t$ is a $G_2$-instanton. Furthermore, $\underline{a}_t$ satisfies $\left|\left|\underline{a}_t\right|\right|_{C^{1,\alpha}_{-1,\delta;t}} \leq ct^{1/18}$.*

We will use this theorem in Section 4.6 to construct a new example of a $G_2$-instanton on the resolution of $(T^3 \times \mathrm{K3})/\mathbb{Z}_2^2$.

## 4.1 The Pregluing Construction

### 4.1.1 Moduli Bundles of ASD-Instantons

Let $\pi : E_0 \to Y/\langle\iota\rangle$ be an orbifold $G$-bundle with connection $\theta$, i.e. a $G$-bundle with connection over $Y$ together with a lift $\hat{\iota}$ of $\iota$ such that $\hat{\iota}^2 = \mathrm{Id}$ and such that $\hat{\iota}^*\theta = \theta$. As before, $\mathrm{fix}(\iota) = L$ and we now set $E_\infty = E_0|_L$, which is a $G$-bundle with $\mathbb{Z}_2$-action, and $A_\infty = \theta|_{E_\infty}$. Denote by $M$ the framed moduli space of ASD instantons on a bundle $E$ over Eguchi-Hanson space $X_{\mathrm{EH}}$ from Section 2.4.2. The homomorphism $\rho : \mathbb{Z}_2 \to G$ used in the definition of $M$ defines a $\mathbb{Z}_2$ left action on $G$. We then ask for $E_0$ and $M$ to be compatible in the following sense:

*Assumption* 4.1. For all $l \in L$ there exists an isomorphism of manifolds with $G$ right action and $\mathbb{Z}_2$ left action $\phi : E_\infty|_l \to G$.

**Proposition 4.2.** *Let $G_\rho \subset G$ be the stabiliser of $\rho$ as in Eq.* (2.43)*. Then there exists a $G_\rho$-reduction $\check{E}$ of $E_\infty$ such that $A_\infty$ reduces to $\check{E}$.*

*Proof.* As before, let $\rho : \mathbb{Z}_2 \to G$ be the representation that defines the asymptotic limit for connections in $M$. Define

$$\check{E} := \{u \in E_\infty : u \cdot \rho(-1) = \hat{\imath}(u)\}. \tag{4.3}$$

To see that this is a $G_\rho$-bundle, fix $l \in L$ and let $\phi : E_\infty|_l \to G$ be the isomorphism from Assumption 4.1. Then $u \in \check{E}|_l$ if and only if $\phi(u) \in G_\rho$.

It remains to check that $A_\infty$ reduces to $\check{E}$. To this end, let $\gamma : I \to \check{E}$ be a curve. Then

$$\begin{aligned} A_\infty(\dot\gamma(0)) &= \hat{\imath}^* A_\infty(\dot\gamma(0)) \\ &= A_\infty\left(\frac{d}{dt}(\gamma(t) \cdot \rho(-1))|_{t=0}\right) \\ &= \mathrm{Ad}(\rho(-1))\left(A_\infty(\dot\gamma(0))\right). \end{aligned} \tag{4.4}$$

In the first step we used $\hat{\imath}^*\theta = \theta$. The second step is the defining property of $\check{E}$ from Eq. (4.3). Now, for any subgroup $H \subset G$ we define the *centraliser of* $H$ *in* $G$ as $Z(H) = \{g \in G : hgh^{-1} = g$ for all $h \in H\}$. Then

$$\mathrm{Lie}(Z(H)) = \mathfrak{z}_H := \{V \in \mathfrak{g} : \mathrm{Ad}(h)V = V \text{ for all } h \in H\}. \tag{4.5}$$

This equality holds, because for $X = \dot g(0) \in \mathrm{Lie}(Z(H))$, where $g : I \to Z(H)$ is a curve, we have that $\mathrm{Ad}(h)X = \frac{d}{dt}(hg(t)h^{-1})|_{t=0} = X$ by definition of $Z(H)$. Conversely, for $V \in \mathfrak{z}_H$, we have that $g(t) := \exp(tV)$ is a curve with $\dot g(0) = V$ in $Z(H)$, because $hg(t)h^{-1} = \exp(t \cdot \mathrm{Ad}(h)V) = \exp(tV) = g(t)$ for all $h \in H$.

Therefore, by Eqs. (4.4) and (4.5), we have that $A_\infty|_{\check{E}}$ takes values in $\mathrm{Lie}(G_\rho)$, i.e. restricts to a connection on $\check{E}$. □

*Definition* 4.6. Define the *moduli bundle*

$$\mathfrak{M} := (\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2) \times G_\rho} M \tag{4.7}$$

and its *vertical tangent space*

$$V\mathfrak{M} := (\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2)\times G_\rho} TM. \tag{4.8}$$

### 4.1.2 Fueter Sections and Connections on Bundles over $P$

In the following, we will study sections $s : L \to \mathfrak{M}$. It will turn out that such a section $s$ gives rise to a connection that is almost a $G_2$-instanton, if it satisfies a first order differential equation, the *Fueter equation* (cf. Definition 4.13).

*Definition* 4.9. Let $s : L \to \mathfrak{M}$ be a section. We define its covariant derivative $\nabla s \in \Omega^1(L, V\mathfrak{M})$ as follows: for $x \in L, X \in T_xL$ let $f \in C^\infty(\mathrm{Fr})$ and $e \in C^\infty(\check{E})$ be local sections around $x$ such that $A^{\mathrm{LC}}\,\mathrm{d}f(x) = 0$ and $A_\infty(\mathrm{d}e(X)) = 0$, where $A^{\mathrm{LC}}$ is the Levi-Civita connection of $Y$. Let $B : L \to M$ be a local section around $x$ such that $s = [(f, e), B]$. Then

$$\nabla_X(s) = [(f, e), dB(X)] \in (\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2)\times G_\rho} TM.$$

*Definition* 4.10. Let $s : L \to \mathfrak{M}$ be a section. Fix $x \in L$ and let $e_1, e_2, e_3$ be an orthonormal basis of $T_xL$. The $G_2$-structure on $Y$ defines a map

$$\begin{aligned} \Lambda^1(T_xL) &\to \Lambda^+ P_x \\ e_i &\mapsto \check{\omega}_i|_{P_x} =: \omega_i. \end{aligned} \tag{4.11}$$

The $\omega_i$ correspond to complex structures on $P_x$ and therefore, by Theorem 2.51, to elements $I_i \in \mathrm{End}(V_x\mathfrak{M})$. We thus have a Clifford multiplication given by

$$\begin{aligned} e_i \cdot : V_x\mathfrak{M} &\to V_x\mathfrak{M} \\ a &\mapsto I_i(a). \end{aligned} \tag{4.12}$$

*Definition* 4.13. A section $s : L \to \mathfrak{M}$ is called a *Fueter section* if

$$\mathfrak{F}s := \sum_{i=1}^{3} e_i \cdot \nabla_{e_i} s = 0 \in \Gamma(s^* V\mathfrak{M}), \tag{4.14}$$

where $(e_1, e_2, e_3)$ is a local orthonormal frame.

The following is an extension of [DS11, Theorem 1]:

**Theorem 4.15.** *Denote by $\widetilde{\mathbb{P}} \to M \times \hat{X}_{EH}$ the tautological bundle with tautological connection $\widetilde{\mathbb{A}}$ over $M \times X_{EH}$ from Proposition 2.59 and assume that there exists a lift of the* $\mathrm{U}(2)$*-action on $M \times \hat{X}_{EH}$ to $\widetilde{\mathbb{P}}$ preserving $\widetilde{\mathbb{A}}$. Let $s \in C^\infty(\mathfrak{M})$, and denote $\hat{P} = \mathrm{Fr} \times_{\mathrm{U}(2)} \hat{X}_{EH}$. Then there exists a natural $G$-bundle $s(E)$ over $\hat{P}$ with connection $s(A) \in \mathscr{A}(s(E)|_P)$ together with an isomorphism of $G$-bundles with $\mathbb{Z}_2$ left action $\Phi : s(E)|_{\hat{P} \setminus P} \to E_\infty$ so that:*

*(i) The pair $(s(E), s(A))|_{P_x}$ represents $s(x)$. That means: if $s(x) = [(f, e), [B]]$ for $f \in \mathrm{Fr}_x$, $e \in (E_0)_x$, $[B] \in M$, then under the diffeomorphism $X_{EH} \simeq P_x$, $y \mapsto [f, y]$, the $G$-bundles $s(E)|_{P_x}$ and $E$ are isomorphic, and $B$ and $s(A)$ are gauge equivalent.*

*(ii) The map $\Phi$ identifies $A_\infty$ and $s(A)$ over the fibre at infinity, i.e. $\Phi^* A_\infty = s(A)|_{\hat{P} \setminus P}$.*

*(iii) The connection $s(A)|_P$ is a $(\psi_t^P)^*$-instanton if and only if $s$ is a Fueter section. Here, $s(A)$ being a $(\psi_t^P)^*$-instanton means that $F_{s(A)} \wedge (\psi_t^P)^* = 0$, where $(\psi_t^P)^* = \sum \sigma^*(e^i) \wedge \sigma^*(e^j) \wedge \check{\omega}^k$. Here $\sigma : P \to L$ is the projection of the bundle Eq.* (3.95)*.*

*Proof.* **Construction of** $s(E)$, $s(A)$, **and** $\Phi$: together with the connections $\nabla^{\mathrm{LC}}$ on Fr and $A_\infty$ on $\check{E}$, the connection $\widetilde{\mathbb{A}}$ induces a connection $\alpha$ on the principal $G$-bundle $(\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2) \times G_\rho} \widetilde{\mathbb{P}} \to (\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2) \times G_\rho} (M \times \hat{X}_{\mathrm{EH}})$ via the formula

$$\alpha([(U, V), T]) := \widetilde{\mathbb{A}}(T), \tag{4.16}$$

where $U \in T\,\mathrm{Fr}$, $V \in T\check{E}$ are horizontal vectors and $T \in T\widetilde{\mathbb{P}}$. By assumption, $\widetilde{\mathbb{A}}$ is left-invariant, which makes the definition of $\alpha$ independent of the chosen representative.

Consider the map

$$\begin{aligned} (s \times \mathrm{Id}) : \hat{P} = \mathrm{Fr} \times_{\mathrm{U}(2)} \hat{X}_{\mathrm{EH}} &\to (\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2) \times G_\rho} (M \times \hat{X}_{\mathrm{EH}}) \\ [f, y] &\mapsto [(f, e), (B, y)], \end{aligned}$$

where $s(\sigma(e)) = [(f, e), B] \in \mathfrak{M}_{\pi(e)}$. Then

$$s(E) := (s \times \mathrm{Id})^*((\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2)\times G_\rho} \widetilde{\mathbb{P}}), \quad s(A) := (s \times \mathrm{Id})^* \alpha$$

and the trivialisation $\underline{\phi} : \widetilde{\mathbb{P}}|_{M^{\mathrm{orb}} \times \{\infty\}} \to G \times M^{\mathrm{orb}}$ from Proposition 2.59 induces an isomorphism

$$\begin{aligned} \Phi : \; & s(E)|_{\hat{P}\setminus P} \\ & \simeq (s \times \mathrm{Id}|_{\hat{X}_{\mathrm{EH}}\setminus X_{\mathrm{EH}}})^* \left( (\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2)\times G_\rho} \widetilde{\mathbb{P}}|_{M\times\{\infty\}} \right) \to s^* \left( (\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2)\times G_\rho} G \times M \right) \\ & \qquad \simeq \check{E} \times_{G_\rho} G \simeq E_\infty. \end{aligned} \tag{4.17}$$

The last point of Proposition 2.59 states that $\underline{\phi}^* A_{\mathrm{product}} = \widetilde{\mathbb{A}}|_{M\times\{\infty\}}$ which implies that $\Phi^* A_\infty = s(A)|_{\hat{P}\setminus P}$.

$s(A)$ **is a** $(\psi_t^P)^*$**-instanton if and only if** $s$ **is a Fueter section**: for easier notation, assume that the bundle Fr is trivial and $\nabla^{\mathrm{LC}}$ is the product connection. The proof of the general case works the same. In this case, $L \times \hat{X}_{\mathrm{EH}} = \hat{P}$ and $s(E) = (s \times \mathrm{Id})^*(\check{E} \times_{G_\rho} \widetilde{\mathbb{P}})$. Then fix $(l, x) \in L \times \hat{X}_{\mathrm{EH}} = \hat{P}$, an orthonormal basis $(e_1, e_2, e_3)$ of $T_l L$ and denote by $(e^1, e^2, e^3)$ its dual basis. Around $l$, write $s(x) = [e, B]$ with the property that $\mathrm{d}e(V)$ is parallel for all $V \in T_l L$. Then, for $Z \in T_x \hat{X}_{\mathrm{EH}}$:

$$\begin{aligned} F_{s(A)}(e_i, Z) &= ((s \times \mathrm{Id})^* F_\alpha)\,(e_i, Z) \\ &= F_\alpha\left([\mathrm{d}e(e_i), (\mathrm{d}B(e_i), 0)], [\mathrm{d}e(e_i), (0, Z)]\right) \\ &= F_{\widetilde{\mathbb{A}}}(\mathrm{d}B(e_i), Z) \\ &= \mathrm{d}B(e_i)(Z). \end{aligned} \tag{4.18}$$

In the first step we used that the curvature of a pullback connection is the pullback of its curvature. The third step is the definition of $\alpha$ from Eq. (4.16), and in the last step we used the curvature properties of the tautological connection $\widetilde{\mathbb{A}}$ from Proposition 2.59. As before, denote by $I_1, I_2, I_3$ the Hyperkähler triple of complex structures on $X_{\mathrm{EH}}$ and $\omega_1, \omega_2, \omega_3$ the corresponding symplectic forms. The Fueter condition from Definition 4.13 for $s$ is equivalent to the following equation of elements in $\Omega^1(X_{\mathrm{EH}}, \mathrm{Ad}\, P)$:

$$0 = \sum_{i=1}^{3} I_i(\mathrm{d}B(e_i)) = \sum_{i=1}^{3} \omega_i(\mathrm{d}B(e_i), \cdot) = \sum_{i=1}^{3} \omega_i(F_{s(A)}(e_i, \cdot), \cdot)$$
$$= * \left( \sum_{i=1}^{3} \omega_i \wedge F_{s(A)}(e_i, \cdot) \right)$$

where $*$ denotes the Hodge star on $X_{\mathrm{EH}}$. The first equality is the Fueter equation, the third equality is Eq. (4.18), and the second and fourth equality are linear algebra computations that can be computed in standard coordinates.

Applying $*$ to both sides gives

$$0 = \left( \sum_{i=1}^{3} \omega_i \wedge F_{s(A)}(e_i, \cdot) \right)$$

which in turn implies

$$0 = \sum_{i,j,k \text{ cyclic}} \omega_i \wedge e^j \wedge e^k \wedge [F_{s(A)}]_{(1,1)},$$

where $[F_{s(A)}]_{(1,1)}$ denotes the $(1,1)$-component of $F_{s(A)}$ according to the bi-grading on $\Lambda^* T^*(L \times X_{\mathrm{EH}})$ induced by $T^*(L \times X_{\mathrm{EH}}) = T^*L \oplus T^*X_{\mathrm{EH}}$. On the other hand, $[F_{s(A)}]_{(0,2)} \in \Omega^2(X_{\mathrm{EH}}, \mathrm{Ad}\, P)$ is anti-self-dual by Proposition 2.59, thus

$$0 = \sum_{i,j,k \text{ cyclic}} \omega_i \wedge e^j \wedge e^k \wedge [F_{s(A)}]_{(0,2)}.$$

Last, $0 = \sum_{i,j,k \text{ cyclic}} \omega_i \wedge e^j \wedge e^k \wedge [F_{s(A)}]_{(2,0)}$, because this is a sum of forms of type $(2,4)$ which must vanish as $L$ has dimension 3. $\square$

### 4.1.3 Gluing Connections over $P$ and $Y/\langle \iota \rangle$

We will define here a further modification of the Hölder norm.

*Definition* 4.19 (cf. Section 6 in [Wal17]). For $\delta, l \in \mathbb{R}$, let

$$
\begin{aligned}
w_{l,\delta;t} : N_t &\to \mathbb{R} \\
x &\mapsto \begin{cases} t^{\delta}(t + r_t(x))^{-l-\delta}, & \text{if } r_t(x) \leq \sqrt{t} \\ r_t^{-l+\delta} & \text{if } r_t(x) > \sqrt{t}. \end{cases}
\end{aligned} \tag{4.20}
$$

Note that $w_{l,\delta;t}$ is not continuous, but that does not cause any problems. For a metric $g$ on $N_t$, define the weighted Hölder norms $||\cdot||_{C^{k,\alpha}_{l,\delta;t},g}$ as in Definition 3.15, where we use parallel transport with respect to the Levi-Civita connection induced by the metric $g$, and measure vectors in $g$. If no metric $g$ is specified, we take $g = g^N_t$. For the instanton analysis, we need $\delta \in (-1, 0)$, $\alpha \in (0, 1)$, $\alpha \ll |\delta|$, for example $\delta = -1/64$, $\alpha = 1/256$ will work.

**Proposition 4.21** (Proposition 6.2 in [Wal17]). *If $(f, g) \mapsto f \cdot g$ is a bilinear form satisfying $|f \cdot g| \leq |f|\,|g|$, then*

$$
||f \cdot g||_{C^{k,\alpha}_{l_1+l_2,\delta_1+\delta_2;t}} \leq ||f||_{C^{k,\alpha}_{l_1,\delta_1;t}} \cdot ||g||_{C^{k,\alpha}_{l_2,\delta_2;t}}.
$$

We have shown that $s(A)$ is a $(\psi^P_t)^*$-instanton. It is, however, in general not a $G_2$-instanton with respect to $\psi^P_t$ because of the $(2, 0)$ part of its curvature. We will later estimate the failure of $s(A)$ of being a $G_2$-instanton.

*Definition* 4.22. For $l \in L$ choose a neighbourhood $l \in V_l \subset L$ over which $E_\infty$ is trivial. Use the identification $\Phi : s(E)|_{\hat{P}\setminus P} \to E_\infty$ and parallel transport with respect to $s(A)$ to get a trivialisation of $s(E)$ around $\hat{P}|_{V_l} \setminus P|_{V_l}$, say on a neighbourhood $U_l \subset \hat{P}$. Using this, we can view the pullback of $s(A)|_{\hat{P}\setminus P}$ under the projection $U_l \to V_l$ as a connection $\overline{A_\infty}^l \in \mathscr{A}(s(E)|_{U_l})$. This definition is independent of the choice of $l \in L$, and therefore defines some connection $\overline{A_\infty} \in \mathscr{A}(s(E)|_U)$, where $U \subset \hat{P}$ is a neighbourhood of the points at infinity $\hat{P} \setminus P$.

Now is the first time we cite a non-trivial result from [Wal17]. Therein, Fueter sections into a moduli bundle of ASD-instantons on $\mathbb{R}^4$ were considered, while in this chapter ASD-instantons on $X_{\mathrm{EH}}$ are considered. At some points this changes the analysis, and these results are reproved in this new setting in the coming sections. At some points, results carry over without having to change the proof. The following proposition is the first such result:

**Proposition 4.23** (Proposition 7.4 in [Wal17]). *There exists $c > 0$ such that for all $t \in (0, T)$:*

$$\left\| [F_{s(A)}]_{2,0} - F_{\overline{A_\infty}} \right\|_{C^{0,\alpha}_{-2,0;t}(U), g^P_t} \leq ct^2, \tag{4.24}$$

$$\left\| [F_{s(A)}]_{1,1} \right\|_{C^{0,\alpha}_{-3,0;t}(U), g^P_t} \leq ct^2, \text{ and} \tag{4.25}$$

$$\left\| [F_{s(A)}]_{0,2} \right\|_{C^{0,\alpha}_{-4,0;t}(U), g^P_t} \leq ct^2. \tag{4.26}$$

**Proposition 4.27.** *Let $E_0 \to Y/\langle\iota\rangle$ be an orbifold bundle with connection $\theta$ satisfying Assumption 4.1, $L = \operatorname{fix}(\iota)$, and $s : L \to \mathfrak{M}$ be a Fueter section.*

*Then there exists a $G$-bundle $E_t$ over $N_t$ and a connection $A_t$ on $E_t$ such that*

$$(E_t, A_t)|_{N_t \setminus \Upsilon_t(U_{t^{-1}R})} \simeq (E_0, \theta)|_{N_t \setminus \Upsilon_t(U_{t^{-1}R})} \quad \text{and}$$
$$(E_t, A_t)|_{\Upsilon_t(U_1)} \simeq (s(E), s(A))|_{\rho^{-1}(U_1)}.$$

*Proof.* **Construction of $E_t$**: By Theorem 4.15 we have a bundle isomorphism $\Phi : E_\infty \to s(E)|_{\hat{P}\setminus P}$. Let $U \subset \hat{P}$ be a neighbourhood of $\hat{P} \setminus P$. Now use radial parallel transport with respect to $\theta$ on $E_0$ and parallel transport with respect to $\overline{A_\infty}$ (the pullback of $\Phi^* A_\infty$ to a neighbourhood of $\hat{P} \setminus P$ defined in Proposition 4.23) to extend $\Phi$ to the neighbourhood $\Upsilon(U_R) \subset Y$ of $L$, denote the extension by $\Psi$. The conditions $\hat{\iota}^*\theta = \theta$ and Assumption 4.1 ensure that this is well-defined.

As in Section 3.3.3 we use the symbol $\rho$ to denote the map $\rho : P \to \nu/\{\pm 1\}$ induced by the blowup map $X_{\mathrm{EH}} \to \mathbb{C}^2/\{\pm 1\}$ on Eguchi-Hanson space. For small enough $t$ we have that the overlap $O := U_{t^{-1}R} \cap \rho(U)$ is non-empty. Use this to define $E_t$ by gluing together $E_0$ and $s(E)$ via $\Psi$ over $O$, i.e.

$$E_t := E_0|_{Y \setminus \Upsilon_t(U_{t^{-1}R} \setminus O)} \cup s(E)|_{\rho^{-1}(U_{t^{-1}R})} /\sim, \tag{4.28}$$

where $v \sim \Psi(v)$ for $v \in E_0|_{\Upsilon_t(O)}$.

**Construction of $A_t$**: Let $\chi^-_t : N_t \to [0, 1]$ and $\chi^+_t : N_t \to [0, 1]$ be rescalings of a smooth

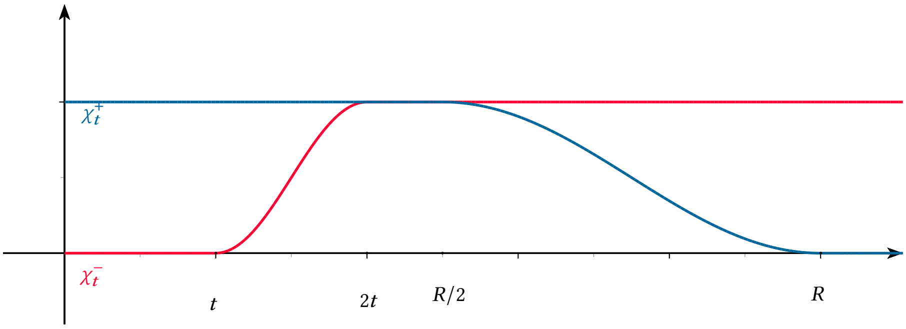


Figure 5: The cut-off functions $\chi_t^-$ and $\chi_t^+$ from Eq. (4.29) for small $t$.

cut-off function such that

$$\begin{aligned} \chi_t^-|_{\{r_t \leq t\}} &\equiv 0 \text{ and } \chi_t^-|_{\{r_t \geq 2t\}} \equiv 1, \\ \chi_t^+|_{\{r_t \leq R/2\}} &\equiv 1 \text{ and } \chi_t^+|_{\{r_t \leq R\}} \equiv 0. \end{aligned} \tag{4.29}$$

Similar to the definition of $\overline{A_\infty} \in \mathscr{A}(s(E)|_U)$, define $\underline{A_\infty} \in \mathscr{A}\left(E_0|_{\Upsilon_t(U_{t^{-1}R})}\right)$ by pulling back $A_\infty \in \mathscr{A}(E_\infty)$. By definition of $E_t$, we have that $\overline{A_\infty}$ and $\underline{A_\infty}$ are both connections on $E_t$. The map $\Phi$ identifies $A_\infty$ and $s(A)$ by the second point of Theorem 4.15. Because $\Psi$ is an extension of $\Phi$ defined by radial parallel transport, and $\overline{A_\infty}$ and $\underline{A_\infty}$ are also defined via radial parallel transport, we have that $\overline{A_\infty} = \underline{A_\infty}$ as connections on $E_t|_{\Upsilon_t(O)}$.

We then have $\sigma \in \Omega^1(\operatorname{Ad} s(E)|_O)$ and $b \in \Omega^1(\operatorname{Ad} E_0|_O)$ such that

$$s(A) = \overline{A_\infty} + \sigma, \qquad \theta = \underline{A_\infty} + b \quad \text{ over } O. \tag{4.30}$$

Define then

$$A_t := \begin{cases} s(A) & \text{on } r_t < t \\ \underline{A_\infty} + \chi_t^- b + \chi_t^+ \sigma & \text{on } t \leq r_t \leq R \\ \theta & \text{on } r_t > R. \end{cases} \tag{4.31}$$

□

The following proposition follows immediately from Definition 4.19.

**Proposition 4.32.** *Let $\chi_t^-$ and $\chi_t^+$ as in Eq.* (4.29). *Then there exists $c > 0$ such that for all $t \in (0, T)$:*

$$\left\|\chi_t^-\right\|_{C^{0,\alpha}_{0,0;t}} + \left\|\mathrm{d}\chi_t^-\right\|_{C^{0,\alpha}_{-1,0;t}} \leq c,$$
$$\left\|\chi_t^+\right\|_{C^{0,\alpha}_{0,0;t}} + \left\|\mathrm{d}\chi_t^+\right\|_{C^{0,\alpha}_{0,0;t}} \leq c.$$

The following proposition is proved like Proposition 4.23 with the proof from [Wal17] directly carrying over to this setting. The estimate for $\sigma$ holds because of the fast decay of the curvature of ASD connections on ALE spaces, see Proposition 2.45. The estimate for $b$ holds because over $L$ we have that $\underline{A_\infty} = \theta$, not just in the $L$-direction. That is because $\underline{A_\infty}$ is defined using parallel transport with respect to $\theta$ as in Definition 4.22.

**Proposition 4.33** (Proposition 7.6 in [Wal17]). *Let $\sigma \in \Omega^1(\operatorname{Ad} s(E)|_O)$ and $b \in \Omega^1(\operatorname{Ad} E_0|_O)$ as defined in Eq.* (4.30). *Then there exists $c > 0$ such that for all $t \in (0, T)$:*

$$||\sigma||_{C^{0,\alpha}_{-3,0;t}(t \leq r_t \leq R)} + \left\|\mathrm{d}_{\overline{A_\infty}}\sigma\right\|_{C^{0,\alpha}_{-4,0;t}(t \leq r_t \leq R)} \leq ct^2 \textit{ and}$$
$$||b||_{C^{0,\alpha}_{1,0;t}(r_t \leq R)} + \left\|\mathrm{d}_{\underline{A_\infty}}b\right\|_{C^{0,\alpha}_{0,0;t}(r_t \leq R)} \leq ct^2.$$

## 4.2 Pregluing Estimate

The goal of this section is to derive an estimate for $F_{A_t} \wedge \widetilde{\psi}_t^N$. This is achieved in Corollary 4.54 in the general case, and in Corollary 4.57 in the special case of resolutions of $T^7/\Gamma$.

### 4.2.1 Estimates for the $G_2$-structures Involved

We have constructed a connection $A_t$ that looks like $s(A)$ near $L$ and looks like $\theta$ far away from $L$. The connection $s(A)$ is close to being a $G_2$-instanton with respect to $\psi_t^P$, so in order to control the pregluing error, we will need to estimate the difference $\psi_t^N - \varphi_t^P$. This will be done in Propositions 4.34 and 4.37.

On the other hand, $\theta$ is a $G_2$-instanton with respect to $\psi$, so we will need to estimate the difference $\psi_t^N - \psi$. This will be done in Proposition 4.39.

**Proposition 4.34.** *There exists $c > 0$ independent of $t$ such that*

$$\left\|\psi_t^N - \psi_t^P\right\|_{C^{0,\alpha}_{2,0;t}(U_R)} \leq ct^{-1}. \tag{4.35}$$

*Proof.* We have

$$\begin{aligned}
&|\psi_t^N - \psi_t^P|_{g_t^N} \\
&= \begin{cases}
\mathrm{d}[t^2\beta_{0,3} + t^4\beta_{2,1}+]t^2\chi_{1,3} + t^4\theta_{3,1} + t^4\theta_{2,2} & \text{if } \check{r} \leq t^{-1/9} \\
\mathrm{d}[t^2\beta_{0,3} + t^4\beta_{2,1} + a(t^{1/9}\check{r}) \cdot \Upsilon_*\zeta] + t^2\chi_{1,3} + t^4\theta_{3,1} + t^4\theta_{2,2} & \text{if } t^{-1/9} \leq \check{r} \leq 2t^{-1/9} \\
\mathrm{d}[t^2\beta_{0,3} + t^4\beta_{2,1} + \Upsilon_*\zeta] + t^2\chi_{1,3} + t^4\theta_{3,1} + t^4\theta_{2,2} & \text{if } 2t^{-1/9} \leq \check{r} \leq t^{-4/5} \\
\begin{aligned}&\mathrm{d}[(1 - a(t^{4/5}\check{r}))(t^2\beta_{0,3} + t^4\beta_{2,1}) + \Upsilon_*\zeta]+ \\ &\quad t^2\chi_{1,3} + t^4\theta_{3,1} + t^4\theta_{2,2} - a(t^{4/5}\check{r})t^2 v_{1,2}\end{aligned} & \text{if } t^{-4/5} \leq \check{r} \leq 2t^{-4/5} \\
\mathrm{d}(\Upsilon_*\zeta) + t^2\chi_{1,3} + t^4\theta_{3,1} + t^4\theta_{2,2} - t^2 v_{1,2} & \text{if } 2t^{-4/5} \leq \check{r}
\end{cases} \qquad (4.36) \\
&= \begin{cases}
O(t) & \text{if } \check{r} \leq t \\
O(t\check{r}^{-3}) & \text{if } t \leq \check{r} \leq t^{-1/9} \\
O(t\check{r}^{-3} + t^2\check{r}^2) & \text{if } t^{-1/9} \leq \check{r} \leq 2t^{-1/9} \\
O(t\check{r}^{-3} + t^2\check{r}^2) & \text{if } 2t^{-1/9} \leq \check{r} \leq t^{-4/5} \\
O(t^2\check{r}^2 + \check{r}^{-4}) & \text{if } t^{-4/5} \leq \check{r} \leq 2t^{-4/5} \\
O(t^2\check{r}^2 + \check{r}^{-4}) & \text{if } 2t^{-4/5} \leq \check{r},
\end{cases}
\end{aligned}$$

where we used Propositions 3.92 and 3.103 and Theorem 3.110 in the second step. Multiplying with the weight function $(t + r_t)^{-2}$ gives the estimate for the $L^\infty_{2,0;t}$-norm, and the estimate for the $C^{0,\alpha}_{2,0;t}$-norm is proved analogously. □

**Proposition 4.37.** *Let $N_t$ be the resolution of $T^7/\Gamma$ from Section 3.2. There exists $c > 0$ independent of $t$ such that*

$$\left\|\psi_t^N - \psi_t^P\right\|_{C^{0,\alpha}_{2,0;t}(U_R)} \leq ct^4. \tag{4.38}$$

*Proof.* This is a restatement of Lemma 3.36. In the case that $N_t$ is the resolution of $T^7/\Gamma$ we have that $\psi_t^P$ is closed, so the forms $t^2\chi_{1,3}$, $t^4\theta_{3,1}$, $t^4\theta_{2,2}$ from Proposition 3.103 can be chosen to be 0. Furthermore, in this case $\widetilde{\psi}_t^\nu = \Upsilon_t^*(*\varphi)$, so $\zeta = 0$. Using this and that the cut-off happens where $\zeta t^{-1}/2 \leq \check{r} \leq \zeta t^{-1}$, the same proof as for Eq. (4.35) shows the claim. □

The following estimate holds in general, not just for resolutions of $T^7/\Gamma$:

**Proposition 4.39.** *There exists* $c > 0$ *independent of* $t$ *such that*

$$\left\|\psi_t^N - \psi\right\|_{C^{0,\alpha}_{-2,0;t}(\{x \in N_t : \check{r}(x) \geq 1\})} \leq ct^2. \tag{4.40}$$

*Proof.* Using Propositions 3.92 and 3.103 and Theorem 3.110, the proof is analogous to Proposition 4.34. □

Last we need an estimate comparing $\widetilde{\psi}_t^N$ and $\psi_t^N$ in a Hölder norm. In Theorem 3.110 we had this estimate for the $L^\infty$-norm, but not for the $C^{0,\alpha}_{0,0;t}$-norm. Going through the proof of 2.26, one can improve this to a $C^{0,\alpha}_{0,0;t}$-estimate as stated in the following proposition. For the case of resolutions of $T^7/\Gamma$, this was done in [Wal13a, Proposition 4.20], and the proof carries over to resolutions of $Y/\langle\iota\rangle$.

**Proposition 4.41.** *There exists* $c > 0$ *independent of* $t$ *such that*

$$\left\|\widetilde{\psi}_t^N - \psi_t^N\right\|_{C^{0,\alpha}_{0,0;t}} \leq ct^{1/18}. \tag{4.42}$$

### 4.2.2 Principal Bundle Curvature Estimates

For our pregluing estimate we will want to estimate $*(F_{A_t} \wedge \widetilde{\psi}_t^N)$. This is done in Corollaries 4.54 and 4.57. Most of the heavy lifting is done by the following Proposition 4.43: here we get an estimate for $*(F_{A_t} \wedge \psi_t^N)$ which then is combined with the estimate for $\widetilde{\psi}_t^N - \psi_t^N$.

**Proposition 4.43.** *There exists* $c > 0$ *such that for all* $t \in (0, T)$ *we have*

$$\left\|*(F_{A_t} \wedge \psi_t^N)\right\|_{C^{0,\alpha}_{-2,0;t}} \leq ct. \tag{4.44}$$

*Proof.* We will estimate $*(F_{A_t} \wedge \psi_t^N)$ separately on some regions:

1. On $r_t \leq 2t$ we have

$$F_{A_t} = F_{s(A)} + \chi_t^- \, \mathrm{d}_{A_\infty} b + \chi_t^- [\sigma, b] + \frac{1}{2}(\chi_t^-)^2 [b, b] + \mathrm{d}\chi_t^- \wedge b.$$

Thus by Proposition 4.21, Proposition 4.32, and Proposition 4.33:

$$\begin{aligned}
&\left\| F_{A_t} - F_{s(A)} \right\|_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \\
&\leq \|1\|_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \left\| \chi_t^- \right\|_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} \left\| \mathrm{d}_{A_\infty} b \right\|_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} \\
&\quad + \left\| \chi_t^- \right\|_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} \|\sigma\|_{C^{0,\alpha}_{-3,0;t}(r_t \leq 2t)} \|b\|_{C^{0,\alpha}_{1,0;t}(r_t \leq 2t)} \\
&\quad + \frac{1}{2} \|1\|_{C^{0,\alpha}_{-3,0;t}(r_t \leq 2t)} \left\| \chi_t^- \right\|^2_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} \|b\|^2_{C^{0,\alpha}_{1,0;t}(r_t \leq 2t)} \\
&\quad + \|1\|_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \left\| \mathrm{d}\chi_t^- \right\|_{C^{0,\alpha}_{-1,0;t}(r_t \leq 2t)} \|b\|_{C^{0,\alpha}_{1,0;t}(r_t \leq 2t)} \\
&\leq ct^2
\end{aligned} \tag{4.45}$$

where we also used the fact that $\|1\|_{C^{0,\alpha}_{-l,0;t}(r_t \leq 2t)} \leq ct^l$ if $l > 0$, which follows from Definition 4.19 using $r_t \leq 2t$.

Remember that $[F_{s(A)}]_{2,0} \wedge \psi_t^P = 0$ by the ASD condition and $[F_{s(A)}]_{1,1} \wedge \psi_t^P = 0$ by the Fueter condition (cf. Theorem 4.15). By Proposition 4.23, we therefore have:

$$\begin{aligned}
&\left\| F_{s(A)} \wedge \psi_t^P \right\|_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \\
&\leq \left\| [F_{s(A)}]_{(0,2)} \wedge \psi_t^P \right\|_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \\
&\leq \left\| [F_{s(A)} - F_{\theta|_L}]_{(0,2)} \right\|_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \cdot \left\| \psi_t^P \right\|_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} + \\
&\quad \left\| F_{\theta|_L} \right\|_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} \cdot \left\| \psi_t^P \right\|_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} \cdot \|1\|_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \\
&\leq ct^2,
\end{aligned} \tag{4.46}$$

where we again used Proposition 4.21. Last, note that by Proposition 4.23 and Eq. (4.45) we have $\left\| F_{A_t} \right\|_{C^{0,\alpha}_{-4,0;t}(r_t \leq 2t)} \leq t^2$ because the weight function in this region is uniformly bounded from above and below by $ct^2$. Thus, by Proposition 4.21 and Eq. (4.35):

$$\begin{aligned} \left\|F_{A_t} \wedge (\psi_t^N - \psi_t^P)\right\|_{C^{0,\alpha}_{-2,0;t}(r_t \le 2t)} &\le \left\|F_{A_t}\right\|_{C^{0,\alpha}_{-4,0;t}(r_t \le 2t)} \left\|\psi_t^N - \psi_t^P\right\|_{C^{0,\alpha}_{2,0;t}(r_t \le 2t)} \\ &\le ct. \end{aligned} \tag{4.47}$$

Putting the estimates from Eqs. (4.45) to (4.47) together, we get

$$\begin{aligned} &\left\|*(F_{A_t} \wedge \psi_t^N)\right\|_{C^{0,\alpha}_{-2,0;t}(r_t \le 2t)} \\ &\le \left\|F_{s(A)} \wedge \psi_t^P)\right\|_{C^{0,\alpha}_{-2,0;t}(r_t \le 2t)} + \left\|(F_{s(A)} - F_{A_t}) \wedge \psi_t^P\right\|_{C^{0,\alpha}_{-2,0;t}(r_t \le 2t)} \\ &\quad + \left\|F_{A_t} \wedge (\psi_t^N - \psi_t^P)\right\|_{C^{0,\alpha}_{-2,0;t}(r_t \le 2t)} \\ &\le c(t^2 + t^2 + t) \le ct. \end{aligned}$$

2. On $2t \le r_t \le R/2$ we have $A_t = A_\infty + \sigma + b$ and therefore

$$F_{A_t} = F_\theta + [\sigma, b] + F_{s(A)} - F_{A_\infty}. \tag{4.48}$$

First,

$$\begin{aligned} &\left\|(F_{s(A)} - F_{A_\infty}) \wedge \psi_t^P\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\ &\le \left\|\left[F_{s(A)} - F_{A_\infty}\right]_{2,0} \wedge \psi_t^P\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\ &\le \left\|\left[F_{s(A)} - F_{A_\infty}\right]_{2,0}\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \left\|\psi_t^P\right\|_{C^{0,\alpha}_{0,0;t}(2t \le r_t \le R/2)} \\ &\le ct^2, \end{aligned} \tag{4.49}$$

where we used point (ii) of Theorem 4.15 in the first step and Proposition 4.23 in the last step. We also have

$$\begin{aligned} &\left\|(F_{s(A)} - F_{A_\infty}) \wedge (\psi_t^N - \psi_t^P)\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\ &\le \left\|(F_{s(A)} - F_{A_\infty})\right\|_{C^{0,\alpha}_{-4,0;t}(2t \le r_t \le R/2)} \left\|\psi_t^N - \psi_t^P\right\|_{C^{0,\alpha}_{2,0;t}(2t \le r_t \le R/2)} \\ &\le ct \end{aligned} \tag{4.50}$$

where we used Proposition 4.23 and Eq. (4.35), therefore

$$\begin{aligned}
&\left\|(F_{s(A)} - F_{A_\infty}) \wedge \psi_t^N\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
\le &\left\|(F_{s(A)} - F_{A_\infty}) \wedge \psi_t^P\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
&+ \left\|(F_{s(A)} - F_{A_\infty}) \wedge (\psi_t^N - \psi_t^P)\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
\le &\, ct.
\end{aligned} \tag{4.51}$$

Second,

$$\begin{aligned}
&\left\|[\sigma, b] \wedge \psi_t^N\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
\le &\, c \left\|\sigma\right\|_{C^{0,\alpha}_{-3,0;t}(2t \le r_t \le R/2)} \left\|b\right\|_{C^{0,\alpha}_{1,0;t}(2t \le r_t \le R/2)} \left\|\psi_t^N\right\|_{C^{0,\alpha}_{0,0;t}(2t \le r_t \le R/2)} \\
\le &\, ct^4
\end{aligned} \tag{4.52}$$

by Proposition 4.33.

Third,

$$\begin{aligned}
&\left\|F_\theta \wedge \psi_t^N\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
\le &\left\|F_\theta \wedge \psi\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
&+ \left\|F_\theta\right\|_{C^{0,\alpha}_{0,0;t}(2t \le r_t \le R/2)} \left\|\psi_t^N - \psi\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
\le &\, ct^2
\end{aligned} \tag{4.53}$$

where we used the fact that $\theta$ is a $G_2$-instanton with respect to $\psi$ as well as Eq. (4.40) in the second step. So, altogether

$$\begin{aligned}
\left\|*(F_{A_t} \wedge \psi_t^N)\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \le &\left\|F_\theta \wedge \psi_t^N\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
&+ \left\|[\sigma, b] \wedge \psi_t^N\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
&+ \left\|(F_{s(A)} - F_{A_\infty}) \wedge \psi_t^N\right\|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
\le &\, ct
\end{aligned}$$

by combining Eqs. (4.48) and (4.51) to (4.53).

3. On $R/2 \le r_t \le R$ we have $A_t = \theta + \chi_t^+ \sigma$ and therefore

$$F_{A_t} = F_\theta + \chi_t^+ \, \mathrm{d}_\theta \sigma + \frac{1}{2}(\chi_t^+)^2 [\sigma, \sigma] + \mathrm{d}\chi_t^+ \wedge \sigma.$$

Therefore, we find that

$$\begin{aligned}
\left\| F_{A_t} - F_\theta \right\|_{C^{0,\alpha}_{-2,0;t}(R/2 \le r_t)} &\le \left\| \chi_t^+ \right\|_{C^{0,\alpha}_{0,0;t}(R/2 \le r_t)} \left|\left| \mathrm{d}_\theta \sigma \right|\right|_{C^{0,\alpha}_{-4,0;t}(R/2 \le r_t)} \left|\left| 1 \right|\right|_{C^{0,\alpha}_{2,0;t}(R/2 \le r_t)} \\
&\quad + \frac{1}{2} \left\| \chi_t^+ \right\|^2_{C^{0,\alpha}_{0,0;t}(R/2 \le r_t)} \left|\left| \sigma \right|\right|^2_{C^{0,\alpha}_{-3,0;t}(R/2 \le r_t)} \left|\left| 1 \right|\right|_{C^{0,\alpha}_{4,0;t}(R/2 \le r_t)} \\
&\quad + \left\| \mathrm{d}\chi_t^+ \right\|_{C^{0,\alpha}_{0,0;t}(R/2 \le r_t)} \left|\left| \sigma \right|\right|_{C^{0,\alpha}_{-3,0;t}(R/2 \le r_t)} \left|\left| 1 \right|\right|_{C^{0,\alpha}_{1,0;t}(R/2 \le r_t)} \\
&\le ct^2
\end{aligned}$$

where we used Propositions 4.21, 4.32 and 4.33 in the second step. Using this, we see

$$\begin{aligned}
\left\| F_{A_t} \wedge \psi_t^N \right\|_{C^{0,\alpha}_{-2,0;t}(R/2 \le r_t)} &\le \left\| (F_{A_t} - F_\theta) \wedge \psi_t^N \right\|_{C^{0,\alpha}_{-2,0;t}(R/2 \le r_t)} \\
&\quad + \left\| F_\theta \wedge \psi_t^N \right\|_{C^{0,\alpha}_{-2,0;t}(R/2 \le r_t)} \\
&\le ct^2,
\end{aligned}$$

where we used the fact that $\psi_t^N = \psi$ where $r_t \ge R/2$ and that $\theta$ is a $G_2$-instanton with respect to $\psi$.

We have that $F_{A_t} \wedge \psi_t^N = 0$ outside the three considered regions, which proves the claim. □

**Corollary 4.54.** *There exists $c > 0$ such that*

$$\left\| *(F_{A_t} \wedge \widetilde{\psi}_t^N) \right\|_{C^{0,\alpha}_{-2,0;t}} \le ct^{1/18}. \tag{4.55}$$

*Proof.* First, observe that

$$\left\| F_{A_t} \right\|_{C^{0,\alpha}_{-2,0;t}} \le c. \tag{4.56}$$

This follows from estimating $F_{A_t}$ separately on the three regions from the proof of Proposition 4.43. Then

$$
\begin{aligned}
\left\|*(F_{A_t} \wedge \widetilde{\psi}_t^N)\right\|_{C^{0,\alpha}_{-2,0;t}} &\leq \left\|*(F_{A_t} \wedge \psi_t^N)\right\|_{C^{0,\alpha}_{-2,0;t}} + \left\|*(F_{A_t} \wedge (\widetilde{\psi}_t^N - \psi_t^N))\right\|_{C^{0,\alpha}_{-2,0;t}} \\
&\leq \left\|*(F_{A_t} \wedge \psi_t^N)\right\|_{C^{0,\alpha}_{-2,0;t}} + \left\|F_{A_t}\right\|_{C^{0,\alpha}_{-2,0;t}} \left\|\widetilde{\psi}_t^N - \psi_t^N\right\|_{C^{0,\alpha}_{0,0;t}} \\
&\leq c(t + t^{1/18}) \leq ct^{1/18}
\end{aligned}
$$

where we used Proposition 4.43 to estimate the first summand in the last step, and Eqs. (4.42) and (4.56) to estimate the second summand in the last step. □

As promised, we now turn to the special case of resolutions of $T^7/\Gamma$, rather than general $G_2$-orbifolds. We get a better pregluing estimate here, which is due to the following two facts: first, we get a better estimate for $*(F_{A_t} \wedge \psi_t^N)$ on the resolution of $T^7/\Gamma$, because near the associative, $A_t$ is close to $s(A)$, which is close to being a $G_2$-instanton with respect to $\psi_t^P$, and Proposition 4.37 says that $\psi_t^N - \psi_t^P$ is small. Second, the difference $\widetilde{\psi}_t^N - \psi_t^N$ is smaller on resolutions of $T^7/\Gamma$ than in the general case.

**Corollary 4.57.** *Let $N_t$ be the resolution of $T^7/\Gamma$ from Section 3.2. Then there exists $c > 0$ such that for all $t \in (0, T)$ we have*

$$
\left\|*(F_{A_t} \wedge \widetilde{\psi}_t^N)\right\|_{C^{0,\alpha}_{-2,0;t}} \leq ct^2. \tag{4.58}
$$

*Proof.* We first prove

$$
\left\|*(F_{A_t} \wedge \psi_t^N)\right\|_{C^{0,\alpha}_{-2,0;t}} \leq ct^2. \tag{4.59}
$$

as in Proposition 4.43, the only difference being that Eq. (4.38) in Eqs. (4.47) and (4.50) gives a factor of $t^2$ rather than $t$, yielding Eq. (4.59). For small enough $\alpha \in (0, 1)$ we have that

$$
\left\|\widetilde{\psi}_t^N - \psi_t^N\right\|_{C^{0,\alpha}_{0,0;t}} \leq ct^{5/2} \tag{4.60}
$$

by Theorem 3.84. Taking Eqs. (4.59) and (4.60) together gives Eq. (4.58) as in the proof of Corollary 4.54. □

## 4.3 Linear Estimates

We now arrived in the second step of the three step process of (1) constructing an approximate solution, (2) estimating the linearisation of the instanton equation, and (3) perturbing the approximate solution to a genuine solution. The estimate in question is Proposition 4.77. It makes use of the norms $||\cdot||_{\mathfrak{X}_t}$ and $||\cdot||_{\mathfrak{Y}_t}$ that are defined in Section 4.3.1, and the analysis is analogous to Section 3.2.3.

The idea of the proof is this: near the resolution locus of the associative $L$, the linearisation of the instanton equation is approximately equal to the linearisation of the Fueter equation. Deformations of the approximate solution and deformations of the Fueter section live in different spaces, so some work will need to go into making this statement precise.

Over the course of Sections 4.3.3 to 4.3.5 we work out an estimate for the linearised operator modulo deformations of the approximate instanton that come from deformations of the Fueter section. This estimate is given in Proposition 4.105. Its proof is very similar to the proof of Proposition 3.65: we use a Schauder estimate for the linearised operator, which is given in section Section 4.3.4, together with analysis on the local models $\mathbb{R}^3 \times X_{\mathrm{EH}}$ and $\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}$, which is explained in Section 4.3.3.

So we have estimates for the linearised operator on instanton deformations that come from deformations of the Fueter section from Section 4.3.2 and on the other instanton deformations from Section 4.3.5. In Sections 4.3.6 and 4.3.7 we combine both and complete the proof of Proposition 4.105.

### 4.3.1 Stating the Estimate

In the previous section, we constructed a connection $A_t \in \mathscr{A}(E_t)$. The linearisation of the $G_2$-instanton equation together with the Coulomb gauge condition is

$$L_t := L_{A_t} : (\Omega^0 \oplus \Omega^1)(M, \operatorname{Ad} E) \to (\Omega^0 \oplus \Omega^1)(M, \operatorname{Ad} E)$$
$$\begin{pmatrix} \xi \\ a \end{pmatrix} \mapsto \begin{pmatrix} 0 & \mathrm{d}^*_{A_t} \\ \mathrm{d}_{A_t} & *(\widetilde{\psi}^N_t \wedge \mathrm{d}_{A_t}) \end{pmatrix} \begin{pmatrix} \xi \\ a \end{pmatrix},$$

cf. Eq. (2.103). We introduce the following notation for the constant part and the quadratic part of the $G_2$-instanton equation: for $\underline{a} = (\xi, a) \in (\Omega^0 \oplus \Omega^1)(N_t, \operatorname{Ad} E_t)$ define $e_t$ as well as $Q_t(\underline{a}) \in \Omega^0(N_t, \operatorname{Ad} E_t)$ via

$$\begin{aligned} &* (F_{A_t+a} \wedge \widetilde{\psi}^N_t) + \mathrm{d}_{A_t+a}\xi \\ &\quad = \underbrace{*(F_{A_t} \wedge \widetilde{\psi}^N_t)}_{=:e_t} + *\,(\mathrm{d}_{A_t} a \wedge \widetilde{\psi}^N_t) + \mathrm{d}_{A_t}\xi + \underbrace{\frac{1}{2} * ([a \wedge a] \wedge \widetilde{\psi}^N_t) + [\xi, a]}_{=:Q_t(\underline{a})}. \end{aligned} \tag{4.61}$$

In this section we will study the operator $L_t$ and derive an estimate for the operator norm of the inverse of $L_t$. This operator norm will be taken with respect to the complicated norms $\|\cdot\|_{\mathfrak{X}}$ and $\|\cdot\|_{\mathfrak{Y}}$, taken from [Wal17, Section 8], which we will explain now.

We need a way to decompose elements in $\Omega^1(N_t, \operatorname{Ad} E_t)$ into a part coming from a section of $s^*(V\mathfrak{M})$, which is nonzero only near the gluing area, and a rest:

*Definition* 4.62. The section $s$ gives rise to a connection $s(A) \in \mathscr{A}(s(E))$ by Theorem 4.15. A section $f \in \Gamma(s^*V\mathfrak{M})$ analogously defines an element in $T_{s(A)}\mathscr{A}(s(E)) = \Omega^1(P, \operatorname{Ad} s(E))$, say $i_* f$. Use this to define

$$\begin{aligned} \iota_t : \Gamma(s^*V\mathfrak{M}) &\to \Omega^1(N_t, \mathfrak{g}_{E_t}) \\ f &\mapsto \chi^+_t \cdot i_* f. \end{aligned} \tag{4.63}$$

Further define $\pi_t : \Omega^1(N_t, \operatorname{Ad} E_t) \to \Gamma(s^* V\mathfrak{M})$ for $a \in \Omega^1(N_t, \operatorname{Ad} E_t)$ and $x \in L$ by

$$(\pi_t a)(x) := \sum_{\kappa} \int_{P_x} \langle a, \iota_t \kappa \rangle_{g_t^P} \operatorname{vol}_{g_t^P|_{P_x}} \cdot \kappa, \tag{4.64}$$

where $\kappa$ runs through an orthonormal basis of $(V\mathfrak{M})_{s(x)}$ with respect to the inner product $\langle \iota_t \cdot, \iota_t \cdot \rangle_{g_P^t}$. Here the integral is taken with respect to the metric induced by $\varphi_t^P$ restricted to $P_x$. Let further $\overline{\pi}_t := \iota_t \pi_t$ and $\eta_t := \operatorname{Id} - \overline{\pi}_t$.

The following proposition states that $\iota_t$ and $\pi_t$ are bounded operators. The proof of these estimates is similar to the proofs of Propositions 3.45 and 3.48 and [Wal17, Proposition 6.4].

**Proposition 4.65.** *For $l \leq -1$ and $\delta \in \mathbb{R}$ such that $l - \alpha + \delta > -3$ and $l + \delta < -1$ there is a constant $c > 0$ such that for all $t \in (0, T)$ we have*

$$\begin{aligned} ||\iota_t f||_{C^{0,\alpha}_{l,\delta;t}} &\leq c t^{-1-l} \, ||f||_{C^{0,\alpha}} \text{ and} \\ ||\pi_t a||_{C^{0,\alpha}} &\leq c t^{1+l-\alpha} \, ||a||_{C^{0,\alpha}_{l,\delta;t}(V_{[0,R),t})} . \end{aligned}$$

*Proof.* The first inequality is proved like Proposition 3.45.

To prove the second inequality, note that by Proposition 2.50 we have for $x \in L, \kappa \in (V\mathfrak{M})_{s(x)}$

$$|i_* \kappa|_{g_1^P} \leq c_\kappa (1 + \check{r})^{-3}$$

for a constant $c_\kappa$ depending on $x \in L$ and on $\kappa$. Because $(V\mathfrak{M})_{s(x)}$ is a finite-dimensional vector space we can take $c = \max_{||\kappa||_{L^2, g_1^P} = 1} c_\kappa$ to get the estimate

$$|i_* \kappa|_{g_1^P} \leq c (1 + \check{r})^{-3} \, ||\kappa||_{g_1^P, L^2} \tag{4.66}$$

for a constant $c$ independent of $\kappa$. By compactness of $L$, we can assume $c$ to also be independent of $x \in L$. By measuring in $g_t^P$ instead of $g_1^P$ we get from Eq. (4.66):

$$|i_* \kappa|_{g_t^P} = t^{-1} |i_* \kappa|_{g_1^P} \leq c t^2 (t + t\check{r})^{-3} \, ||\kappa||_{g_1^P, L^2} . \tag{4.67}$$

For some interval $J \subset \mathbb{R}$ and $x \in L$ we denote $P_{x,J} := \{u \in P_x : \check{r}(u) \in J\}$ and similarly for

$(\nu/\{\pm1\})_{x,J}$. By abuse of notation we write $\mathrm{vol}_{g_t^P}$ for $\mathrm{vol}_{g_t^P|_{P_x}} \in \Omega^4(P_x)$ and similarly for $\mathrm{vol}_{g_t^\nu}$.

$$
\begin{aligned}
\int_{P_x} \langle a, \chi_t^+ \cdot i_*\kappa\rangle_{g_t^P} \mathrm{vol}_{g_t^P} \leq& \int_{P_x} |a|_{g_t^P} |\chi_t^+ \cdot i_*\kappa|_{g_t^P} \mathrm{vol}_{g_t^P} \\
&\leq c \int_{P_{x,[0,1]}} \frac{t^2}{(t+t\check{r})^3} w_{l,\delta;t}^{-1} \mathrm{vol}_{g_t^P} \; ||a||_{L^\infty_{l,\delta;t}, g_t^P} ||\kappa||_{L^2, g_1^P} \\
&\quad + c \int_{P_{x,[1,Rt^{-1}]}} \frac{t^2}{(t+t\check{r})^3} w_{l,\delta;t}^{-1} \mathrm{vol}_{g_t^P} \; ||a||_{L^\infty_{l,\delta;t}, g_t^P} ||\kappa||_{L^2, g_1^P} \\
&\leq c \, \mathrm{vol}_{g_t^P}(P_x, [0,1]) \cdot t^{l-1} \; ||a||_{L^\infty_{l,\delta;t}, g_t^P} ||\kappa||_{L^2, g_1^P} \\
&\quad + c \int_{(\nu/\{\pm1\})_{x,[0,Rt^{-1}]}} \frac{t^2}{(t+t\check{r})^3} w_{l,\delta;t}^{-1} \mathrm{vol}_{g_t^\nu} \; ||a||_{L^\infty_{l,\delta;t}, g_t^P} ||\kappa||_{L^2, g_1^P} \\
&\leq ct^{l+3} \; ||a||_{L^\infty_{l,\delta;t}, g_t^P} ||\kappa||_{L^2, g_1^P} \\
&\quad + c \int_0^{\sqrt{t}} t^{2-\delta}(t+r)^{l+\delta-3} r^3 \, \mathrm{d}r \cdot ||a||_{L^\infty_{l,\delta;t}, g_t^P} ||\kappa||_{L^2, g_1^P} \\
&\quad + c \int_{\sqrt{t}}^{R} t^2 r^{l-\delta}(t+r)^{-3} r^3 \, \mathrm{d}r \cdot ||a||_{L^\infty_{l,\delta;t}, g_t^P} ||\kappa||_{L^2, g_1^P} .
\end{aligned}
\tag{4.68}
$$

Here we used Eq. (4.67) in the second step. In the third step, we switched from integrating over $P_{x,[1,Rt^{-1}]}$ to integrating over $\nu_{x,[1,Rt^{-1}]}$. We could do this because $t\check{r}$ on $P$ corresponds to the radius function $r$ on $\nu$, and $g_t^P|_{P_{x,[1,Rt^{-1}]}} - \rho^* g_t^\nu|_{P_{x,[1,Rt^{-1}]}} \longrightarrow 0$ measured in $g_t^\nu$ as $t \longrightarrow 0$ by Eqs. (3.93) and (3.100). The latter implies that we can change $\mathrm{vol}_{g_t^P}$ to $\mathrm{vol}_{g_t^\nu}$ by Proposition A.4. We used the definition of $w_{l,\delta;t}$ and changing into sphere coordinates in the fourth step.

We now treat the two integrals separately.

$$
\begin{aligned}
\int_0^{\sqrt{t}} (t+r)^{l+\delta-3} r^3 \, \mathrm{d}r &= \Bigg[ (r+t)^{\delta+l} \Bigg( -\frac{3t}{\delta+l} - \frac{t^3}{(-2+\delta+l)(r+t)^2} \\
&\qquad + \frac{3t^2}{(-1+\delta+l)(r+t)} + \frac{r+t}{1+\delta+l} \Bigg) \Bigg]_0^{\sqrt{t}} \\
&\leq c(t^{\delta+l+1} + t^{\delta/2+l/2+1/2}) \\
&\leq ct^{\delta+l+1},
\end{aligned}
\tag{4.69}
$$

where we used a computer algebra system to compute the integral in the first step and used $\delta + l + 1 < 0$ in the third step. For the second integral we find that

$$
\begin{aligned}
\int_{\sqrt{t}}^{R} r^{l-\delta}(t+r)^{-3} r^3 \,\mathrm{d}r &\leq \int_{\sqrt{t}}^{R} r^{l+1-\delta} \,\mathrm{d}r \\
&\leq \left[ r^{l+1-\delta} \right]_{\sqrt{t}}^{R} \\
&\leq t^l \cdot t^{-l/2-\delta/2-1/2} \cdot t^1 + c \\
&\leq c t^{l+1}
\end{aligned}
\tag{4.70}
$$

where we used the fact that $-l - \delta - 1 > 0$ to estimate the first summand in the last step, and the fact that $l \leq -1$ to estimate the second summand in the last step.

Combining Eqs. (4.68) to (4.70) we get

$$
\int_{P_x} \langle a, \chi_t \cdot i_* \kappa \rangle_{g_t^P} \operatorname{vol}_{g_t^P} \leq c t^{3+l} \, ||a||_{L^\infty_{l,\delta;t}} \, ||\kappa||_{L^2, g_1^P} \,. \tag{4.71}
$$

If $\kappa_1, \kappa_2 \in (V\mathfrak{M}_t)_{s(x)}$, then

$$
\begin{aligned}
\langle \chi_t^+ \cdot i_* \kappa_1, \chi_t^+ \cdot i_* \kappa_2 \rangle_{L^2, g_t^P} &\sim \langle i_* \kappa_1, i_* \kappa_2 \rangle_{L^2, g_t^P} \\
&\sim t^2 \langle i_* \kappa_1, i_* \kappa_2 \rangle_{L^2, g_1^P},
\end{aligned}
\tag{4.72}
$$

where $\sim$ means comparable uniformly in $t$. Here, in the second step we used the fact that $\operatorname{vol}_{g_t^P|_{P_x}} = t^4 \operatorname{vol}_{g_1^P|_{P_x}}$ and $\langle \kappa_1(y), \kappa_2(y) \rangle_{g_t^P} = t^{-2} \langle \kappa_1(y), \kappa_2(y) \rangle_{g_1^P}$ for $y \in P_x$. Equation (4.72) implies that if $\kappa$ has unit length with respect to the inner product $\langle \iota_t \cdot, \iota_t \cdot \rangle_{g_t^P}$, then

$$
||\kappa||_{L^2, g_1^P} \leq c t^{-1}. \tag{4.73}
$$

Because $||\cdot||_{L^2, g_1^P}$ and $||\cdot||_{L^\infty, g_1^P}$ are norms on a finite-dimensional vector space, they are equivalent, and thus

$$
||\kappa||_{L^\infty, g_1^P} \leq c t^{-1}. \tag{4.74}
$$

Combining Eqs. (4.71), (4.73) and (4.74) and recalling the definition of $\pi_t$ from Definition 4.62

gives

$$
\begin{aligned}
||\pi_t a||_{L^\infty} &\leq \left| \sum_\kappa \int_{P_x} \langle a, \iota_t \kappa \rangle_{g_t^P} \operatorname{vol}_{g_t^P|_{P_x}} \right| \cdot ||\kappa||_{L^\infty, g_1^P} \\
&\leq c t^{1+l} \, ||a||_{L^\infty_{l,\delta;t}} .
\end{aligned}
$$

The estimate for the $||\cdot||_{C^{0,\alpha}}$ Hölder norm follows analogously. □

We are now ready to define the norms which we will use to prove estimates for the operator $L_t$:

*Definition* 4.75. Denote by $\mathfrak{X}_t$ and $\mathfrak{Y}_t$ the Banach spaces $C^{1,\alpha}(N_t, (\Lambda^0 \oplus \Lambda^1) \otimes \operatorname{Ad} E_t)$ and $C^{0,\alpha}(N_t, (\Lambda^0 \oplus \Lambda^1) \otimes \operatorname{Ad} E_t)$ equipped with the norms

$$
\begin{aligned}
\left\|\underline{a}\right\|_{\mathfrak{X}_t} &:= t^{-\delta/2} \left\|\eta_t \underline{a}\right\|_{C^{1,\alpha}_{-1,\delta;t}} + t \left\|\pi_t \underline{a}\right\|_{C^{1,\alpha}} \quad \text{and} \\
\left\|\underline{a}\right\|_{\mathfrak{Y}_t} &:= t^{-\delta/2} \left\|\eta_t \underline{a}\right\|_{C^{0,\alpha}_{-2,\delta;t}} + t \left\|\pi_t \underline{a}\right\|_{C^{0,\alpha}}
\end{aligned}
\tag{4.76}
$$

respectively.

Using these norms, we can now state the main result of this section:

**Proposition** 4.77. *Let $N_t$ be the resolution of $T^7/\Gamma$ from Section 3.2. Let $s$ be the Fueter section and $\theta$ be the $G_2$-instanton used in the construction of $A_t$ (cf. Proposition 4.27). If $s$ is infinitesimally rigid and $\theta$ is infinitesimally rigid and irreducible, then there exists a constant $c > 0$ which is independent of $t$ such that for small enough $t$ and all $\underline{a} \in (\Omega^0 \oplus \Omega^1)(N_t, \operatorname{Ad} E_t)$:*

$$
\left\|\underline{a}\right\|_{\mathfrak{X}_t} \leq c \left\|L_t \underline{a}\right\|_{\mathfrak{Y}_t} . \tag{4.78}
$$

Unfortunately, we are restricted to the case where $N_t$ is a resolution of $T^7/\Gamma$. The reason for this is that in this case we have improved control over the $G_2$-structure $\widetilde{\varphi}_t^N$ as proved in Proposition 4.37 and Theorem 3.84. The proof of the proposition extends over the rest of this section.

### 4.3.2 Comparison with the Fueter Operator

Given an element $v \in \Gamma(s^*V\mathfrak{M})$ one may do two different things to it: either embed it into $\Omega^1(N_t, \operatorname{Ad} E_t)$ first, and then apply $L_t$. Or apply the linearised Fueter operator first, and then embed it into $\Omega^1(N_t, \operatorname{Ad} E_t)$.

Compare this situation with Section 3.2.3: there we considered an element in $\Omega^0(L)$ and could either embed it into $\Omega^2(N_t)$ first, and then apply $\Delta_{N_t}$. Or we could apply $\Delta_L$ first, and then embed it into $\Omega^2(N_t)$. In that case it turned out that the two are the same up to a small error, cf. Proposition 3.55.

In this new situation this still turns out the be true with a similar proof. In [Wal17], Fueter sections into a moduli bundle of ASD-instantons on $\mathbb{R}^4$ were considered, and the following proposition was proved in that setting. In this chapter ASD-instantons on $X_{\mathrm{EH}}$ are considered, but the proof works essentially the same way. That said, we do need that $\widetilde{\psi}_t^N - \psi_t^P$ is small. This is true on resolutions of $T^7/\Gamma$ by Proposition 4.37 and Theorem 3.84 but not proved for general resolutions of $G_2$-orbifolds. Consequently, we only know the following two propositions to hold on resolutions of $T^7/\Gamma$.

**Proposition 4.79** (Proposition 8.26 in [Wal17]). *Let $N_t$ be the resolution of $T^7/\Gamma$ from Section 3.2. There exists a constant $c > 0$ such that for all $t \in (0, T)$ and all $v \in \Gamma(s^*V\mathfrak{M})$ the following estimate holds:*

$$||L_t \iota_t v - \iota_t \, \mathrm{d}_s \mathfrak{F} v||_{C^{0,\alpha}_{-2,0;t}} \leq ct^2 \, ||v||_{C^{1,\alpha}} . \tag{4.80}$$

The following proposition is a consequence of Proposition 4.79 that is proved like Proposition 3.61. It essentially provides the estimate for the inverse of $L_t$ on the space $\operatorname{Im} \overline{\pi}_t \subset \Omega^1(N_t, \operatorname{Ad} E_t)$.

**Proposition 4.81.** *Let $N_t$ be the resolution of $T^7/\Gamma$ from Section 3.2. If $s$ is infinitesimally rigid, then there exists a constant $c > 0$ such that for all $t \in (0, T)$ and all $v \in \Gamma(s^*V\mathfrak{M})$ the following* *estimate holds:*

$$||v||_{C^{1,\alpha}} \leq c\,||\pi_t L_t \iota_t v||_{C^{0,\alpha}}\,. \tag{4.82}$$

### 4.3.3 The Model Operators on $\mathbb{R}^3 \times X_{\mathrm{EH}}$ and $\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}$

As before, let $X_{\mathrm{EH}}$ be the Eguchi-Hanson space. To prove the estimate in Proposition 4.77, we will compare the operator $L_t$ with the linearised instanton equation in the model case of a pulled back ASD instanton on $\mathbb{R}^3 \times X_{\mathrm{EH}}$.

**Properties of the Model Operator**

Let $A$ be a finite energy ASD instanton on a $G$-bundle $E$ over $X_{\mathrm{EH}}$. The infinitesimal deformations of $A$ are then governed by the operator $\delta_A$ from Eq. (2.31). Denote by $p_{X_{\mathrm{EH}}} : \mathbb{R}^3 \times X_{\mathrm{EH}} \to X_{\mathrm{EH}}$ the projection onto the second factor. By a slight abuse of notation we denote the pullbacks of $A$ and $E$ to $\mathbb{R}^3 \times X_{\mathrm{EH}}$ under $p_{X_{\mathrm{EH}}}$ by $A$ and $E$ as well.

Denote by $L_A$ be the linearised $G_2$-instanton operator from Eq. (2.105). We can define the map $(\,\cdot\,)^\sharp \lrcorner \varphi : p^*_{\mathbb{R}^3} T^*\mathbb{R}^3 \xrightarrow{\simeq} p^*_{X_{\mathrm{EH}}} \Lambda^+ T^* X_{\mathrm{EH}}$, which takes a 1-form, dualises it, and plugs it into the product $G_2$-structure $\varphi$ from Eq. (2.27). It maps $\mathrm{d}x_i$ to $-\omega_i$. Using it, we can relate $\delta_A$ and $L_A$ as follows:

**Proposition 4.83** (Proposition 2.70 in [Wal13b]). *Under the identification*

$$(\,\cdot\,)^\sharp \lrcorner \varphi : p^*_{\mathbb{R}^3} T^*\mathbb{R}^3 \xrightarrow{\simeq} p^*_{X_{EH}} \Lambda^+ T^* X_{EH}$$

*and accordingly*

$$\Omega^0 \oplus \Omega^1(\mathbb{R}^3 \times X_{EH}, \operatorname{Ad} E) \simeq \Omega^0(\mathbb{R}^3 \times X_{EH}, p^*_{X_{EH}}[(\mathbb{R} \oplus \Lambda^+ T^* X_{EH} \oplus T^* X_{EH}) \otimes \operatorname{Ad} E])$$

*the operator $L_A$ can be written as $L_A = F + D_A$ where*

$$F(\xi, \omega, a) = \sum_{i=1}^{3} (-\langle \partial_i \omega, \omega_i \rangle, \partial_i \xi \cdot \omega_i, I_i \partial_i a) \quad \text{and} \quad D_A = \begin{pmatrix} 0 & \delta_A \\ \delta_A^* & 0 \end{pmatrix}.$$

*Moreover,*

$$L_A^* L_A = \Delta_{\mathbb{R}^3} + \begin{pmatrix} \delta_A \delta_A^* & \\ & \delta_A^* \delta_A \end{pmatrix}. \tag{4.84}$$

Recall the weighted Hölder norms $||\cdot||_{C^{0,\alpha}_\beta}$ on $\mathbb{R}^3 \times X_{\mathrm{EH}}$ from Definition 3.40. The following proposition is then a consequence of Lemma 3.38:

**Proposition 4.85** (Proposition 2.74 in [Wal13b]). *Let $\widetilde{X}$ be an ALE space. Let $\beta \in (-3, 0)$. Then $\underline{a} \in C^{1,\alpha}_\beta$ is in the kernel of $L_I : C^{1,\alpha}_\beta \to C^{0,\alpha}_{\beta-1}$ if and only if it is given by the pullback of an element of the $L^2$ kernel of $\delta_I$ to $\mathbb{R}^3 \times \widetilde{X}$.*

**Comparison with $L_t$**

We now explain two maps $s^P$ and $s^\nu$: the first for "zooming into" the resolution locus of the associative $L$, the second for "zooming into" the gluing region of $N_t$. Fix a point $y \in L$, a scaling parameter $d \in \mathbb{Z}$, a gluing parameter $t \in (0, T)$, and two positive real numbers $\epsilon_1, \epsilon_2$ defining the scale of the region into which to zoom in.

Let

$$\begin{aligned} V^P_{\epsilon_1,\epsilon_2;t}(y) &:= \{x \in P : \sigma(x) \in \mathrm{Im}(\exp_y|_{(-\epsilon_1,\epsilon_1)^3}), \check{r}(x)t < \epsilon_2\} \subset P, \\ U^P_{\epsilon_1/t,\epsilon_2/t;t} &:= \{(x,z) \in \mathbb{R}^3 \times X_{\mathrm{EH}} : x \in (-\epsilon_1/t, \epsilon_1/t)^3, \rho(z) < \epsilon_2/t\}. \end{aligned}$$

Here we implicitly used an identification $T_yL \simeq \mathbb{R}^3$ to have $\exp_y$ acting on $(-\epsilon_1, \epsilon_1)^3$. Choose this identification so that it maps the orthonormal basis $e_1(y), e_2(y), e_3(y) \in T^*_yL$ from Section 3.3.3 to the standard basis $\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3 \in \Lambda^1((\mathbb{R}^3)^*)$. Fix an element $f \in \mathrm{Fr}_y$ of the unitary frame bundle of $\nu$ around $y \in L$. It induces an isometry $X_{\mathrm{EH}} \simeq P_y$, and assume that $f$ is chosen so that $\omega_i$ is sent to $\check{\omega}_i|_{P_y}$ under this map for $i \in \{1,2,3\}$. Then, for small $\epsilon_1$, we define

$$\begin{aligned} E^P : U^P_{\epsilon_1/t,\epsilon_2/t;t} &\to V^P_{\epsilon_1,\epsilon_2;t}(y) \\ (x,z) &\mapsto \mathcal{P}_{s\mapsto \exp_y(tsx)}(f(z)) \in P. \end{aligned} \tag{4.86}$$

Here, $s \mapsto \exp_y(sx)$ denotes the unique shortest geodesic from $y$ to $\exp(tx)$ in $L$, and $\mathcal{P}_{s\mapsto\exp_y(tsx)}$ denotes parallel transport in $P$ with respect to $\check{H}$ along this curve, cf. the paragraph before Eq. (3.98). For $\epsilon_1$ small enough, this is a diffeomorphism. The reason for this definition is the following: because of our choices of identifications $T_yL \simeq \mathbb{R}^3$ and $P_y \simeq X_{\mathrm{EH}}$ we have that $(E^P)^*(\varphi_t^P)(0,z)$ is the standard $G_2$-structure on $\mathbb{R}^3 \times X_{\mathrm{EH}}$, for all $z \in X_{\mathrm{EH}}$, cf. Eq. (3.98). Let $a$ be a tensor field of valence $(p,q)$, i.e. in index notation $p$ lower indices and $q$ upper indices, on $V^P_{\epsilon_1,\epsilon_2;t}(y)$. We then define

$$s^P(a) := s^{P,\epsilon_1,\epsilon_2}_{d,y;t}(a) := t^{d+p-q}(E^P)^*a, \tag{4.87}$$

which is a tensor on $U_{\epsilon_1/t,\epsilon_2/t;t}$. The point of this is the following proposition:

**Proposition 4.88.** *There are constants $c > 0$ and $\epsilon > 0$ such that for small $t$ the following holds: for all $\epsilon_1, \epsilon_2 \in (0,\epsilon)$ and for all $\underline{a} \in (\Omega^0 \oplus \Omega^1)(N_t, E_t)$:*

$$\left\| s^{P,\epsilon_1,\epsilon_2}_{d,t;y}\underline{a} \right\|_{L^\infty_{l+\delta}(U^P_{\epsilon_1/t,\epsilon_2/t;t})} \sim t^{d+l}\left\|\underline{a}\right\|_{L^\infty_{l,\delta;t}(V^P_{\epsilon_1,\epsilon_2}(y))}, \tag{4.89}$$

$$\left\| s^{P,\epsilon_1,\epsilon_2}_{d,t;y}\underline{a} \right\|_{C^{k,\alpha}_{l+\delta}(U^P_{\epsilon_1/t,\epsilon_2/t;t})} \sim t^{d+l}\left\|\underline{a}\right\|_{C^{k,\alpha}_{l,\delta;t}(V^P_{\sqrt{t},\sqrt{t}}(y))}, \tag{4.90}$$

*where $\sim$ means comparable independently of $t$. Furthermore, using the Hyperkähler isomorphism $P_y \simeq X_{EH}$ induced by $f$, we can view the connection $s(A)$ over $P_y$ as a connection over $X_{EH}$, denoted by $f_*(s(y))$. Then*

$$\left\| L_t\underline{a} - \left(s^{P,\sqrt{t},\sqrt{t}}_{2,t;y}\right)^{-1} L_{p^*_{X_{EH}}f_*(s(y))} s^{P,\sqrt{t},\sqrt{t}}_{1,t;y}\underline{a} \right\|_{C^{0,\alpha}_{-2,\delta;t}(V^P_{\sqrt{t},\sqrt{t}}(y))} \leq c\sqrt{t}\left\|\underline{a}\right\|_{C^{1,\alpha}_{-1,\delta;t}(V^P_{\sqrt{t},\sqrt{t}}(y))}. \tag{4.91}$$

*Proof.* We first prove Eq. (4.89): for $(0,z) \in U_{\epsilon_1/t,\epsilon_2/t;t}$ the map $\mathrm{d}_{(0,z)}E^P$ (cf. Eq. (4.86)) is an isometry for the metric $t^2(g_{\mathbb{R}^3} \oplus g_{(1)})$ on $T_{(0,z)}(\mathbb{R}^3 \times X_{\mathrm{EH}})$ and the metric on $T_{E^P(0,z)}P$ induced by $g_t^P$. Because of the scaling factor $t^{d+p-q}$ from Eq. (4.96) we have that

$$|s^{P,\epsilon_1,\epsilon_2}_{d,t;y}\underline{a}(0,z)|_{g_{\mathbb{R}^3}\oplus g_{(1)}} = t^d|\underline{a}(E^P(0,z))|_{g_t^P}. \tag{4.92}$$

The map $E^P$ is not, in general, an isometry away from this one point, as $\exp_y$ need not be an isometry. Thus, Eq. (4.92) need not hold in points different from $(0,z)$. However, using Taylor expansions in a neighbourhood of $y$ in $L$ for $\underline{a}$ and $g_t^P$ we get

$$\left\| s^{P,\epsilon_1,\epsilon_2}_{d,t;y} \underline{a} \right\|_{L^\infty_{l+\delta}(U_{\epsilon_1/t,\epsilon_2/t;t})} \sim t^{d+l} \left\| \underline{a} \right\|_{L^\infty_{l,\delta;t}(V_{\epsilon_1,\epsilon_2}(y)),g_t^P} .$$

Now Eq. (4.36) and Proposition 4.41 give the claim for the metric $\widetilde{g}_t^N$ instead of $g_t^P$, which is Eq. (4.89). Equation (4.90) is proved analogously.

Now to prove Eq. (4.91): as in Eq. (4.92), we see that for $x \in P_y$, $\check{r}(x) < 1/\sqrt{t}$,

$$L_{s(A)}\underline{a}(x) - \left( \left( s^{P,\sqrt{t},\sqrt{t}}_{2,t;y} \right)^{-1} L_{p^*_{X_{\mathrm{EH}}} f_*(s(y))} s^{P,\sqrt{t},\sqrt{t}}_{1,t;y} \underline{a} \right)(x) = 0. \tag{4.93}$$

And $A_t - s(A) = O(1)$ on $P_y$, so

$$\begin{aligned}
&\left\| L_t \underline{a} - \left( \left( s^{P,\sqrt{t},\sqrt{t}}_{2,t;y} \right)^{-1} L_{p^*_{X_{\mathrm{EH}}} f_*(s(y))} s^{P,\sqrt{t},\sqrt{t}}_{1,t;y} \underline{a} \right) \right\|_{C^{0,\alpha}_{-2,\delta;t}(\{u \in P_y : \check{r}(u) < 1/\sqrt{t}\})} \\
&\leq c \left|\left| [A_t - s(A), a] \right|\right|_{C^{0,\alpha}_{-2,\delta;t}(\{u \in P_y : \check{r}(u) < 1/\sqrt{t}\})} \\
&\leq c \left|\left| a \right|\right|_{C^{0,\alpha}_{-1,\delta;t}(\{u \in P_y : \check{r}(u) < 1/\sqrt{t}\})} \left|\left| A_t - s(A) \right|\right|_{C^{0,\alpha}_{-1,0;t}(\{u \in P_y : \check{r}(u) < 1/\sqrt{t}\})} \\
&\leq c\sqrt{t} \left|\left| a \right|\right|_{C^{0,\alpha}_{-1,\delta;t}(\{u \in P_y : \check{r}(u) < 1/\sqrt{t}\})} \\
&\leq c\sqrt{t} \left\| \underline{a} \right\|_{C^{1,\alpha}_{-1,\delta;t}(\{u \in P_y : \check{r}(u) < 1/\sqrt{t}\})}
\end{aligned} \tag{4.94}$$

where in the third step we used $A_t - s(A) = O(1)$ to estimate the second factor as $\sqrt{t}$. This was possible because the weight function is bounded by $\sqrt{t}$ on $\{u \in P_y : \check{r}(u) < 1/\sqrt{t}\}$.

Equation (4.91) now follows from using Taylor expansions for $\underline{a}$, $g_t^P$, and $s$ around $y$, and comparing $g_t^P$ and $\widetilde{g}_t^N$ as in the proof of Eq. (4.89). □

We now define $s^\nu$: let $\epsilon_1 > 0$, $\epsilon_2 > \epsilon_3 > 0$, and

$$\begin{aligned}
V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3;t}(y) &:= \{x \in \nu/\{\pm 1\} : \sigma(x) \in \mathrm{Im}(\exp_y|_{(-\epsilon_1,\epsilon_1)^3}), \epsilon_3 < r(x) < \epsilon_2\}, \\
U^\nu_{\epsilon_1/t,\epsilon_2/t,\epsilon_3/t;t} &:= \{(x,z) \in \mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\} : x \in (-\epsilon_1/t, \epsilon_1/t)^3, \epsilon_3/t < |\rho(z)| < \epsilon_2/t\}.
\end{aligned}$$

Just as in the definition of $V^P_{\epsilon_1,\epsilon_2;t}$, we implicitly used an identification $T_yL \simeq \mathbb{R}^3$ so that $e^i$ is sent to $\mathrm{d}x^i$ for $i \in \{1,2,3\}$. Recall also the frame $f$ that sends $\omega_i$ to $\check{\omega}_i|_{P_y}$ for $i \in \{1,2,3\}$ under the isometry $X_{\mathrm{EH}} \simeq P_y$ induced by $f$. We see from Eq. (3.97) that $\omega_i^{(0)}$ is sent to $\hat{\omega}_i|_{v_y}$ under the isometry $\mathbb{C}^2/\{\pm 1\} \simeq (\nu/\{\pm 1\})_y$ induced by $f$. For small $\epsilon_1, \epsilon_2, \epsilon_3$, the map

$$\begin{aligned} E^\nu : U^\nu_{\epsilon_1/t,\epsilon_2/t,\epsilon_3/t;t} &\to V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3;t}(y) \\ (x,z) &\mapsto \mathcal{P}^\nu_{s\mapsto \exp_y(tsx)}(f(z)) \in \nu/\{\pm 1\} \end{aligned} \tag{4.95}$$

is a diffeomorphism, where $\mathcal{P}^\nu$ denotes parallel transport in $\nu$ with respect to the connection $\widetilde{\nabla}^\nu$ from Proposition 3.92. Because of our choices of identifications $T_yL \simeq \mathbb{R}^3$ and $(\nu/\{\pm 1\})_y \simeq \mathbb{C}^2/\{\pm 1\}$ we have that $(E^P)^*(\varphi_t^\nu)(0,z)$ is the standard $G_2$-structure on $\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}$, for all $z \in \mathbb{C}^2/\{\pm 1\}$, cf. Eq. (3.96). We now define $s^\nu$ just as we defined $s^P$ in Eq. (4.96), only exchanging $E^P$ for $E^\nu$: for a tensor field $a$ of valence $(p,q)$ on $V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3;t}(y)$ set

$$s^\nu(a) := s^{\nu,\epsilon_1,\epsilon_2,\epsilon_3}_{d,y;t}(a) := t^{d+p-q}(E^\nu)^* a. \tag{4.96}$$

In the following we use the norms from Definition 3.40. So, the notation $C^{0,\alpha}_0$ does not mean zero boundary condition, but means that the weight function appears with powers of $0$ and $0+\alpha$ in the two summands of the definition $||\cdot||_{C^{0,\alpha}_0}$. We have the following analogue of Proposition 4.88:

**Proposition 4.97.** *There are constants $c > 0$ and $\epsilon > 0$ such that for small $t$ the following holds: for all $\epsilon_1, \epsilon_2 \in (0,\epsilon)$, $\epsilon_3 \in (t,\epsilon)$ and for all $\underline{a} \in (\Omega^0 \oplus \Omega^1)(N_t, E_t)$:*

$$\left\| w^\nu_{l,\delta;t} s^{\nu,\epsilon_1,\epsilon_2,\epsilon_3}_{d,t;y} \underline{a} \right\|_{L^\infty_0(U^\nu_{\epsilon_1/t,\epsilon_2/t,\epsilon_3/t;t})} \sim t^{d+l} \left\| \underline{a} \right\|_{L^\infty_{l,\delta;t}(V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3}(y))}, \tag{4.98}$$

$$\left\| w^\nu_{l,\delta;t} s^{\nu,\epsilon_1,\epsilon_2,\epsilon_3}_{d,t;y} \underline{a} \right\|_{C^{k,\alpha}_0(U^\nu_{\epsilon_1/t,\epsilon_2/t,\epsilon_3/t;t})} \sim t^{d+l} \left\| \underline{a} \right\|_{C^{k,\alpha}_{l,\delta;t}(V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3}(y))}, \tag{4.99}$$

*where $\sim$ means uniformly comparable in $t$ and*

$$w^\nu_{l,\delta;t} = \begin{cases} r^{-l-\delta} & \text{if } r \le 1/\sqrt{t} \\ r^{-l+\delta} t^\delta & \text{if } r > 1/\sqrt{t}. \end{cases}$$

*Furthermore, using the Hyperkähler isomorphism $P_y \simeq X_{EH}$ induced by $f$, we can view the connection $s(A)$ over $P_y$ as a connection over $X_{EH}$. By Eqs. (2.41) and (2.43), this connection is asymptotic* *to a flat connection, say $A_0$, on the orbifold $\mathbb{C}^2/\{\pm 1\}$ with monodromy representation $\rho$. Then*

$$\begin{aligned}&\left\|L_t\underline{a}-\left(s^{\nu,\epsilon_1,\epsilon_2,\epsilon_3}_{2,t;y}\right)^{-1}L_{p^*_{\mathbb{C}^2}A_0}s^{\nu,\epsilon_1,\epsilon_2,\epsilon_3}_{1,t;y}\underline{a}\right\|_{C^{0,\alpha}_{-2,\delta;t}(V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3}(y))}\\ &\le c(\epsilon_1+\epsilon_2+(t/\epsilon_3)^2)\left\|\underline{a}\right\|_{C^{1,\alpha}_{-1,\delta;t}(V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3}(y))},\end{aligned}\tag{4.100}$$

*where $p_{\mathbb{C}^2}:\mathbb{R}^3\times\mathbb{C}^2/\{\pm 1\}\to\mathbb{C}^2/\{\pm 1\}$ denotes the projection onto the second factor.*

*Proof.* Equations (4.98) and (4.99) are proved as in Proposition 4.88.

We now prove Eq. (4.100). Adapting Eq. (4.94) to the area $\{u\in P_y:\epsilon_3/t<\check{r}(u)<\epsilon_2/t\}$ we get

$$\begin{aligned}&\left\|L_t\underline{a}-\left(\left(s^{P,\epsilon_1,\epsilon_2}_{2,t;y}\right)^{-1}L_{p^*_{X_{\mathrm{EH}}}f_*(s(y))}s^{P,\epsilon_1,\epsilon_2}_{1,t;y}\underline{a}\right)\right\|_{C^{0,\alpha}_{-2,\delta;t}(\{u\in P_y:\epsilon_3/t<\check{r}(u)<\epsilon_2/t\})}\\ &\le c\epsilon_2\left\|\underline{a}\right\|_{C^{1,\alpha}_{-1,\delta;t}(\{u\in P_y:\epsilon_3/t<\check{r}(u)<\epsilon_2/t\})}.\end{aligned}\tag{4.101}$$

We have $\left\|p^*_{X_{\mathrm{EH}}}f_*(s(y))-A_0\right\|_{C^{0,\alpha}_{0;0}}=O((\rho\circ p_{X_{\mathrm{EH}}})^{-2})$ by Eq. (2.43) and the fact that we use $\delta=-2$ in our definition of moduli space (cf. Proposition 2.45). Thus, for $x\in P_y$ with $\epsilon_3/t<\check{r}(x)t<R$,

$$\left|\left(s^{P,\sqrt{t},\sqrt{t}}_{2,t;y}\right)^{-1}\left[L_{p^*_{X_{\mathrm{EH}}}f_*(s(y))}-L_{p^*_{X_{\mathrm{EH}}}A_0}\right]s^{P,\sqrt{t},\sqrt{t}}_{1,t;y}\underline{a}\right|_{\widetilde{g}^N_t}(x)\le c(t/\epsilon_3)^2.\tag{4.102}$$

Combining Eqs. (4.101) and (4.102) we get the desired Eq. (4.100) on $P_y\cap V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3}(y)$. Equation (4.100) then follows like Eq. (4.91) by taking Taylor expansions of $\underline{a}$, $g^P_t$, and $s$ around $y$, and this time comparing $g^\nu_t$ and $\widetilde{g}^N_t$ using Eq. (3.93) and Propositions 3.99, 4.34 and 4.41. □

### 4.3.4 Schauder Estimate

On $Y/\langle\iota\rangle$ we have the estimate

$$\left\|\underline{a}\right\|_{C^{1,\alpha}}\le c\left(\left\|L_\theta\underline{a}\right\|_{C^{0,\alpha}}+\left\|\underline{a}\right\|_{L^\infty}\right)$$

from standard elliptic theory, e.g. [Bes87, Section H]. With some additional work, we get an estimate for weighted norms on $\mathbb{R}^3\times X_{\mathrm{EH}}$ (see [Wal17, Proposition 8.15]), and can then glue these two estimates together to obtain:

**Proposition 4.103** (Proposition 8.15 in [Wal17]). *There exists $c > 0$ such that for all $t \in (0, T)$ the following estimate holds:*

$$\left\|\underline{a}\right\|_{C^{1,\alpha}_{-1,\delta;t}} \leq c \left( \left\|L_t \underline{a}\right\|_{C^{0,\alpha}_{-2,\delta;t}} + \left\|\underline{a}\right\|_{L^{\infty}_{-1,\delta;t}} \right). \tag{4.104}$$

### 4.3.5 Estimate of $\eta_t \underline{a}$

The following proposition is the crucial ingredient in the construction of solutions to the instanton equation:

**Proposition 4.105.** *There exists a constant $c > 0$ independent of $t$ such that for $t$ small enough and for all $\underline{a} \in (\Omega^0 \oplus \Omega^1)(N_t, \operatorname{Ad} E_t)$ the following estimate holds:*

$$||a||_{L^{\infty}_{-1,\delta;t}} \leq c \left( \left\|L_t \underline{a}\right\|_{C^{0,\alpha}_{-2,\delta;t}} + \left\|\overline{\pi}_t \underline{a}\right\|_{L^{\infty}_{-1,\delta;t}} \right). \tag{4.106}$$

*Proof.* Assume not, then there exist $t_i \to 0$ and $\underline{a}_i$ such that

$$\left\|\underline{a}_i\right\|_{L^{\infty}_{-1,\delta;t_i}} \equiv 1, \tag{4.107}$$

$$\lim_{i \to \infty} \left\|L_{t_i} \underline{a}\right\|_{C^{0,\alpha}_{-2,\delta;t_i}} = 0, \tag{4.108}$$

$$\lim_{i \to \infty} \left\|\overline{\pi}_{t_i} \underline{a}\right\|_{L^{\infty}_{-1,\delta;t_i}} = 0. \tag{4.109}$$

It follows from Proposition 4.103 that

$$\left\|\underline{a}_i\right\|_{C^{1,\alpha}_{-1,\delta;t}} \leq c. \tag{4.110}$$

Let $x_i \in N_{t_i}$ such that

$$w_{-1,\delta;t}(x_i) \left|\underline{a}_i\right| (x_i) = 1. \tag{4.111}$$

Without loss of generality we can assume to be in one of three following cases, and we will arrive at a contradiction in each of them.

**Case 1.** "$\underline{a}_i$ goes to zero near $L$ and on the neck", i.e. if $z_i \in N_{t_i}$ such that $r_{t_i}(z_i) \to 0$, then $w_{-1,\delta;t}(z_i) \left|\underline{a}_i\right|(z_i) \to 0$.

Without loss of generality, the sequence $(x_i)$ accumulates away from $L$, i.e. $\lim_{i\to\infty} r_{t_i}(x_i) > 0$ (see Fig. 6).

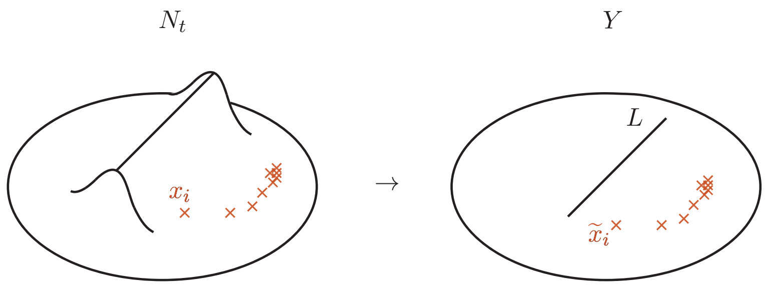


Figure 6: Blowup analysis away from the associative is reduced to the analysis of $\theta$ on $Y$.

Without loss of generality assume that $x_i \to x^* \in Y/\langle\iota\rangle$, where we used that $(Y\setminus L)/\langle\iota\rangle \subset N_{t_i}$, cf. Definition 3.111. Now, using a diagonal argument and the Arzelà–Ascoli theorem, we find that a subsequence of $\underline{a}_i$ converges to a limit $\underline{a}^* \in \Omega^1((Y\setminus L)/\langle\iota\rangle, \mathrm{Ad}\, E_0)$ in $C^{1,\alpha/2}_{\mathrm{loc}}$. Denote by $\pi_\iota : Y \to Y/\langle\iota\rangle$ the quotient map, and denote by $\widetilde{x}_i$ an arbitrary lift of $x_i$, i.e. $\pi_\iota(\widetilde{x}_i) = x_i$. By passing to a subsequence we still have $\widetilde{x}_i \to \widetilde{x}^*$ for some $\widetilde{x}^* \in Y$. Denote also $\widetilde{\underline{a}}^* := \pi_\iota^* \underline{a}^* \in (\Omega^0 \oplus \Omega^1)(\mathrm{Ad}\, E_0|_{Y\setminus L})$.

Equation (4.108) implies that this limit satisfies $L_\theta \widetilde{\underline{a}}^* = 0$ on $Y\setminus L$. We can extend $\widetilde{\underline{a}}^*$ to all of $Y$ as a distribution, and we find that then $L_\theta \widetilde{\underline{a}}^* = 0$ on $Y$ in the sense of distributions. By elliptic regularity, e.g. [Fol95, Theorem 6.33], we have that $\widetilde{\underline{a}}^*$ is smooth.

Lastly, we note that Eq. (4.111) implies $\widetilde{\underline{a}}^*(\widetilde{x}^*) \neq 0$. By assumption, $\theta$ is infinitesimally rigid and irreducible, which is a contradiction.

**Case 2.** "The sequence does not go to zero near $L$", i.e. there exists $y_i \in N_{t_i}$ such that $t_i^{-1} r_{t_i}(y_i) \nrightarrow \infty$, but $w_{-1,\delta;t}(y_i) \left|\underline{a}_i\right|(y_i) \nrightarrow 0$.

Without loss of generality assume that this is the sequence $(x_i)$, i.e. $\lim_{i\to\infty} t_i^{-1} r_{t_i}(x_i) < \infty$ (see Fig. 7).

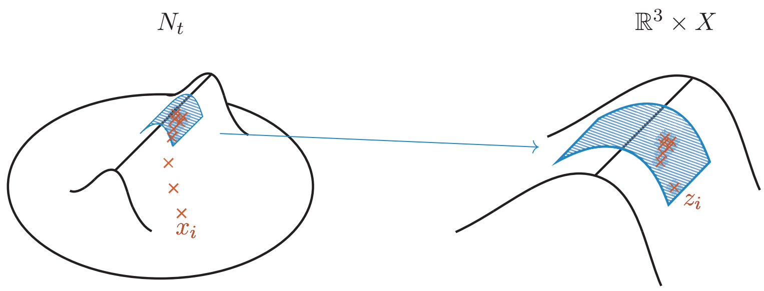


Figure 7: Blowup analysis near the associative is, by means of the map $s^P$, reduced to the analysis of the pull-back of the ASD instanton defined by $s(\sigma(y^*))$ to $\mathbb{R}^3 \times X_{\mathrm{EH}}$.

For $\underline{a}_i = (\xi_i, a_i) \in (\Omega^0 \oplus \Omega^1)(N_t, \mathrm{Ad}\, E_t)$, let

$$\underline{b}_i := \left( s^{P,\sqrt{t_i},\sqrt{t_i}}_{1,\sigma(x_i);t_i}(\xi_i), s^{P,\sqrt{t_i},\sqrt{t_i}}_{1,\sigma(x_i);t_i}(a_i) \right).$$

Proposition 4.88 then gives

$$\left\| \underline{b}_i \right\|_{C^{1,\alpha}_{-1+\delta}(U^P_{1/\sqrt{t_i},1/\sqrt{t_i}})} \leq c \text{ and } \lim_{i \to \infty} \left\| L_{p^*_{X_{\mathrm{EH}}} f_* s(\sigma(x_i))} \underline{b}_i \right\|_{C^{0,\alpha}_{-2+\delta}} = 0.$$

Without loss of generality we can assume $\sigma(x_i) \to y^* \in L$. By a diagonal argument and the Arzelà–Ascoli theorem, we have $\underline{b}_i \to \underline{b}^* \in (\Omega^0 \oplus \Omega^1)(\mathbb{R}^3 \times X_{\mathrm{EH}}, \mathrm{Ad}\, p^*_{X_{\mathrm{EH}}} f_* s(\sigma(y^*)))$ in $C^{1,\alpha/2}_{\mathrm{loc}}$, satisfying $L_{p^*_{X_{\mathrm{EH}}} f_* s(\sigma(y^*))} \underline{b}^* = 0$. Proposition 4.85 implies that $\underline{b}^* = p^*_{X_{\mathrm{EH}}} \underline{c}$, for some $\underline{c} \in \mathrm{Ker}\, L_{f_* s(\sigma(y^*))} \subset \Omega^1(X_{\mathrm{EH}}, f_* s(\sigma(y^*)))$. Equation (4.109) implies that $\underline{c} = 0$ like in Case 1 in the proof of Proposition 3.65.

This contradicts Eq. (4.111) as follows: denote by $(z_i) \subset \mathbb{R}^3 \times X_{\mathrm{EH}}$ the sequence corresponding to $(x_i)$ under the map $s^{\sqrt{t},1/\sqrt{t}}_{1,t_i;\sigma(x_i)}$. Then $(z_i)$ is a bounded sequence, as the $\mathbb{R}^3$-coordinate of all $z_i$ is 0, and the $X_{\mathrm{EH}}$-coordinates are bounded by the assumption that $\lim_{i \to \infty} t_i^{-1} r_{t_i}(x_i) < \infty$. Thus we can assume without loss of generality that $z_i \to z^* \in \mathbb{R}^3 \times X_{\mathrm{EH}}$, and find that

$$w(z^*)^{1-\delta} \left| \underline{b}^*(z^*) \right| = \lim_{i \to \infty} w^{\nu}_{l,\delta;t}(z_i)^{1-\delta} \left| \underline{b}_i(z_i) \right| \geq \frac{1}{c}$$

by Proposition 4.88, which is a contradiction to $\underline{b}^* = 0$.

**Case 3.** "The sequence does not go to zero on the neck", i.e. there exists $y_i \in N_{t_i}$ such that $r_{t_i}(y_i) \to 0$, $t_i^{-1} r_{t_i}(y_i) \to \infty$, but $w_{-1,\delta;t}(y_i) \left| \underline{a}_i \right| (y_i) \nrightarrow 0$.

Assume without loss of generality that this is the sequence $(x_i)$, i.e. $\lim_{i\to\infty} t_i^{-1} r_{t_i}(x_i) = \infty$ and $\lim_{i\to\infty} r_{t_i}(x_i) = 0$ (see Fig. 8).

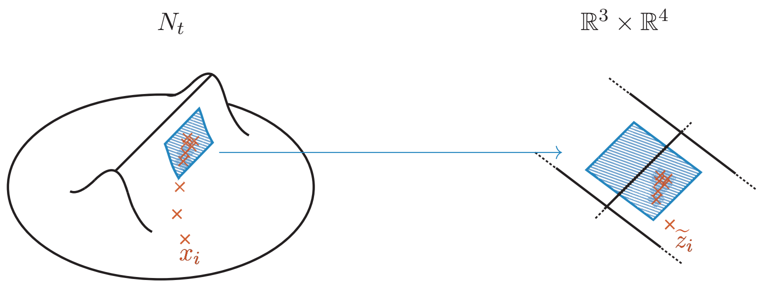


Figure 8: Blowup analysis in the neck region is reduced to the analysis of the flat $G_2$-instanton defined on the pull-back of the framing at infinity defined by $s(\sigma(y^*))$ to $\mathbb{R}^3 \times \mathbb{R}^4$.

Let

- $\epsilon_2^{(i)}$ such that $\epsilon_2^{(i)} \to 0$ and $\epsilon_2^{(i)}/r_{t_i}(x_i) \to \infty$,
- $\epsilon_3^{(i)}$ such that $\epsilon_3^{(i)}/r_{t_i}(x_i) \to 0$ and $\epsilon_3^{(i)}/t_i \to \infty$.

To ease notation, we write $\epsilon_2$ instead of $\epsilon_2^{(i)}$ and $\epsilon_3$ instead of $\epsilon_3^{(i)}$ in what follows. As before, write $\underline{a}_i = (\xi_i, a_i) \in (\Omega^0 \oplus \Omega^1)(N_t, \operatorname{Ad} E_t)$, and set

$$\underline{b}_i := (\zeta_i, b_i) := \left( s_{1,\sigma(x_i);t_i}^{\nu,\sqrt{t_i},\epsilon_2,\epsilon_3}(\xi_i), s_{1,\sigma(x_i);t_i}^{\nu,\sqrt{t_i},\epsilon_2,\epsilon_3}(a_i) \right)$$

and denote by $(z_i)$ the sequence in $\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}$ corresponding to $(x_i)$ under the map $s_{1,\sigma(x_i);t_i}^{\nu,\sqrt{t_i},\epsilon_2,\epsilon_3}$. Equation (4.111) implies

$$|\underline{b}_i(z_i)| \cdot w(z_i) > c, \tag{4.112}$$

Proposition 4.97 and Eq. (4.110) imply that

$$\left\| w_{l,\delta;t}^{\nu} s_{d,t;y}^{\nu,\epsilon_1,\epsilon_2,\epsilon_3} \underline{a} \right\|_{C_0^{1,\alpha}(U^{\nu}_{1/\sqrt{t},\epsilon_2/t,\epsilon_3/t;t})} \leq c, \tag{4.113}$$

Proposition 4.97 and Eq. (4.108) imply that

$$\left\| w_{l,\delta;t}^{\nu} L_{p_{X_{\mathrm{EH}}}^* A_0} s_{1,t;y}^{\nu,\epsilon_1,\epsilon_2,\epsilon_3} \underline{a} \right\|_{C_0^{1,\alpha}(U^{\nu}_{1/\sqrt{t},\epsilon_2/t,\epsilon_3/t;t})} \to 0 \text{ as } i \to \infty. \tag{4.114}$$

We will now conclude the argument as in case 2. The only difference is that, as it stands, the points $z_i$ tend to infinity. Because of this, we cannot directly conclude that a limit of $\underline{b}_i$ would be non-zero. That is why we rescale by $|z_i|$ first. By passing to a subsequence we can assume without loss of generality to be in case 3.1 or 3.2 as below:

**Case 3.1.**: $|z_i| \leq 1/\sqrt{t_i}$. In this case let

$$\underline{\widetilde{b}}_i := (\widetilde{\zeta}_i, \widetilde{b}_i) := \left( |z_i|^{1-\delta} (\cdot |z_i|)^* \zeta_i, |z_i|^{-\delta} (\cdot |z_i|)^* b_i \right). \tag{4.115}$$

Equation (4.112) implies $|\underline{\widetilde{b}}_i(z_i/|z_i|)| \cdot r^{1-\delta}(z_i/|z_i|) = |\underline{\widetilde{b}}_i(z_i/|z_i|)| > c$, and Eq. (4.113) implies that on the sets $B^3(0, 1/\sqrt{t}) \times [B^4(0, \epsilon_2/|x_i|) \setminus B^4(0, \epsilon_3/|x_i|)]$, which exhaust $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\} \setminus \{0\})$, we have:

$$\left\| \begin{cases} \underline{\widetilde{b}}_i r^{1-\delta} & \text{if } r \leq 1/(\sqrt{t} \cdot |z_i|) \\ \underline{\widetilde{b}}_i r^{1+\delta} t^\delta |z_i|^{2\delta} & \text{if } r > 1/(\sqrt{t} \cdot |z_i|). \end{cases} \right\|_{C^{1,\alpha}_0(B^3(0,1/\sqrt{t}) \times [B^4(0,\epsilon_2/|x_i|) \setminus B^4(0,\epsilon_3/|x_i|)])} \leq c. \tag{4.116}$$

Here is how to arrive at the exponents of the weight function for $\widetilde{\zeta}_i$ in the area $\{(u, v) \in \mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\} : r(v) > 1/(\sqrt{t} \cdot |z_i|)\}$:

$$\begin{aligned} \widetilde{\zeta}_i r^{1+\delta} t^\delta |z_i|^{2\delta} &= (\cdot |z_i|)^* \zeta_i |z_i|^{1+\delta} r^{1+\delta} t^\delta \\ &= (\cdot |z_i|)^* \left[ \zeta_i r^{1+\delta} t^\delta \right], \end{aligned}$$

and $\zeta_i r^{1+\delta} t^\delta$ was bounded by Eq. (4.113). The exponents of the weight function on the area $\{(u, v) \in \mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\} : r(v) > 1/(\sqrt{t} \cdot |z_i|)\}$ and also for the 1-form part $\widetilde{b}_i$ are computed analogously and precisely give Eq. (4.116). Now, because of Eq. (4.116), we can use the Arzelà-Ascoli theorem and a diagonal sequence argument to extract a limit $\underline{b}^*$ on $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\} \setminus \{0\})$. We denote the pullback under the quotient map $\mathbb{R}^3 \times (\mathbb{C}^2 \setminus \{0\}) \to \mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\} \setminus \{0\})$ by the same symbol and end up with a tensor $\underline{b}^*$ on $\mathbb{R}^3 \times (\mathbb{C}^2 \setminus \{0\})$. Again, by passing to a subsequence we can assume without loss of generality that we are in one of the following two cases:

**Case 3.1.1:** $\sqrt{t_i}|z_i| \to 0$ as $i \to \infty$.

In this case, the area $\{u \in \mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\} : r(u) > 1/(\sqrt{t} \cdot |z_i|)\}$ disappears as $i \to \infty$, and from Eq. (4.116) we get the estimate

$$\left\| \underline{b}^* r^{1-\delta} \right\|_{C_0^{1,\alpha/2}(\mathbb{R}^3 \times (\mathbb{R}^4 \setminus \{0\}))} \leq c. \tag{4.117}$$

The element $\underline{b}^*$ defines a distribution on all of $\mathbb{R}^3 \times \mathbb{C}^2$ and is smooth by elliptic regularity, e.g. [Fol95, Theorem 6.33]. As in the proof of Proposition 3.65, we get an $L^\infty$-bound for $\underline{b}^*$. Thus, by Corollary 3.39, we get that $\underline{b}^*$ is independent of the $\mathbb{R}^3$-direction. Because of Eq. (4.84) we have that $\underline{b}^*$ is the pullback of a harmonic form of mixed degree (in degrees 0 and 1) on $\mathbb{C}^2$. So, $\underline{b}^*$ is harmonic and bounded on $\mathbb{C}^2$ by Eq. (4.117), therefore vanishes by Liouville's theorem. That contradicts Eq. (4.112).

**Case 3.1.2:** $\sqrt{t_i}|z_i| \to \kappa \in (0, \infty)$ as $i \to \infty$.

In this case, after passing to a subsequence, Eq. (4.116) gives the estimate

$$\left\| \begin{cases} \underline{b}^* r^{1-\delta} & \text{if } r \leq 1/\kappa \\ \underline{b}^* r^{1+\delta} & \text{if } r > 1/\kappa. \end{cases} \right\|_{C_0^{1,\alpha}(\mathbb{R}^3 \times (\mathbb{C}^2 \setminus \{0\})} \leq c. \tag{4.118}$$

Here is how to obtain this estimate: the assumption $\sqrt{t_i}|z_i| \to \kappa$ implies that $\sqrt{t_i}|z_i| > c$, at least up to a subsequence. Thus, we have $t^\delta \cdot |z_i|^{2\delta} < c$, and Eq. (4.116) becomes

$$\left\| \begin{cases} \widetilde{\underline{b}}_i r^{1-\delta} & \text{if } r \leq 1/(\sqrt{t} \cdot |z_i|) \\ \widetilde{\underline{b}}_i r^{1+\delta} & \text{if } r > 1/(\sqrt{t} \cdot |z_i|). \end{cases} \right\|_{C_0^{1,\alpha}(B^3(0,1/\sqrt{t}) \times [B^4(0,\epsilon_2/|x_i|) \setminus B^4(0,\epsilon_3/|x_i|)])} \leq c.$$

Here, taking the limit $i \to \infty$ gives Eq. (4.118). In this case, we arrive at a contradiction as in case 3.1.1.

**Case 3.2.**: $|z_i| > 1/\sqrt{t_i}$. In this case let

$$\widetilde{\underline{b}}_i := (\widetilde{\zeta}_i, \widetilde{b}_i) := \left( t^\delta |z_i|^{1+\delta} (\cdot |z_i|)^* \zeta_i, t^\delta |z_i|^\delta (\cdot |z_i|)^* b_i \right). \tag{4.119}$$

This gives us the following analogue of Eq. (4.116):

$$\left\| \begin{cases} \underline{\widetilde{b}}_i r^{1-\delta} t^{-\delta} |z_i|^{-2\delta} & \text{if } r \leq 1/(\sqrt{t} \cdot |z_i|) \\ \underline{\widetilde{b}}_i r^{1+\delta} & \text{if } r > 1/(\sqrt{t} \cdot |z_i|). \end{cases} \right\|_{C_0^{1,\alpha}(B^3(0,1/\sqrt{t}) \times [B^4(0,\epsilon_2/|x_i|) \setminus B^4(0,\epsilon_3/|x_i|)])} \leq c. \quad (4.120)$$

We can extract a limit $\underline{b}^*$ as in case 3.1 and are, without loss of generality, in one of the following two cases:

**Case 3.2.1:** $\sqrt{t_i} \cdot |z_i| \to \infty$ as $i \to \infty$. In this case we have the estimate

$$\left\| \underline{b}^* r^{1+\delta} \right\|_{C_0^{1,\alpha/2}(\mathbb{R}^3 \times (\mathbb{R}^4 \setminus \{0\}))} \leq c \quad (4.121)$$

and arrive at a contradiction as in case 3.1.1.

**Case 3.2.2:** $\sqrt{t_i} \cdot |z_i| \to \kappa \in (0, \infty)$ as $i \to \infty$. In this case we have exactly Eq. (4.118) and can conclude the proof as in case 3.1.2. □

### 4.3.6 Cross-term Estimates

In the beginning of Section 4.3 we explained the idea for the proof of the linear estimate. Namely, we want to separately consider two parts of the linearisation of the instanton equation: the first part near the resolution locus of the associative $L$, which is approximately equal to the linearisation of the Fueter equation. The second part is the linearised operator modulo deformations of the Fueter section. These parts were estimated in Sections 4.3.2 and 4.3.5.

However, it is not true that the linearised instanton operator neatly decomposes as a sum of these two operators. We can take a deformation of the Fueter section, apply $L_t$ to it, and then project it onto the part that does *not* come from a deformation of the Fueter section. In an ideal world, $L_t$ near the resolution locus of the associative is exactly equal to the linearisation of the Fueter equation and the result is 0. In reality, we do not have that the result is 0, but we have that it is small. That is Eq. (4.123). There is also, roughly speaking, the converse of this, which is Eq. (4.124).

This proposition is the analogue of Proposition 3.77 from the estimate of the Laplacian on the Generalised Kummer Construction. A crucial difference between the present case, i.e. Proposition 4.122, and Proposition 3.77 is that we now get a worse cross-term estimate for $\pi_t L_t \eta_t$. For the Laplacian, we had a factor of roughly $t^2$, while now we have a factor of roughly 1. The reason for this is that $\pi_t$ and $\Delta$ are very close to commuting. The reason they do not exactly commute is because of a cut-off that happens far away from $L$. For the linearised instanton operator $L_t$ the situation is different: the connection $A_t$ was defined to look like $\theta$ already very close to $L$. Thus, $\pi_t L_t$ and $\mathrm{d}_s \mathfrak{F} \pi_t$ are far from being equal, which manifests itself in this worse estimate.

Like the results from Section 4.3.2, this proposition has been proved in a slightly different setting in [Wal17]. Again, the proof given therein carries over to our situation if we only have that $\widetilde{\psi}_t^N - \psi_t^P$ is small, which is true on resolutions of $T^7/\Gamma$ by Proposition 4.37 and Theorem 3.84.

**Proposition 4.122** (Proposition 8.29 in [Wal17]). *Let $N_t$ be the resolution of $T^7/\Gamma$ from Section 3.2. There exists a constant $c > 0$ such that for all $t \in (0, T)$ we have*

$$||\eta_t L_t \iota_t v||_{C^{0,\alpha}_{-2,0;t}} \leq c t^{2-\alpha} ||v||_{C^{1,\alpha}} \tag{4.123}$$

*as well as*

$$\left\|\pi_t L_t \eta_t \underline{a}\right\|_{C^{0,\alpha}} \leq c t^{-\alpha} \left\|\eta_t \underline{a}\right\|_{C^{1,\alpha}_{-1,0;t}}. \tag{4.124}$$

### 4.3.7 Proof of Proposition 4.77

*Proof.* Assume that the claim does not hold, and let $t_i \to 0$, $\underline{a}_i \in (\Omega^0 \oplus \Omega^1)(N_t, \mathrm{Ad}\, E_t)$ such that $\left\|\underline{a}_i\right\|_{\mathfrak{X}_t} = 1$, but $\left\|L_t \underline{a}_i\right\|_{\mathfrak{Y}_t} \to 0$.

We first prove that

$$t_i^{-\delta/2} \left\|\eta_{t_i} \underline{a}_i\right\|_{C^{1,\alpha}_{-1,\delta;t_i}} \to 0. \tag{4.125}$$

We have that

$$
\begin{aligned}
\left\|\eta_{t_i}\underline{a}_i\right\|_{C^{1,\alpha}_{-1,\delta;t_i}} &\leq \left\|L_{t_i}\eta_{t_i}\underline{a}_i\right\|_{C^{0,\alpha}_{-2,\delta;t_i}} \\
&\leq \left\|\eta_{t_i}L_{t_i}\eta_{t_i}\underline{a}_i\right\|_{C^{0,\alpha}_{-2,\delta;t_i}} + \left\|\overline{\pi}_{t_i}L_{t_i}\eta_{t_i}\underline{a}_i\right\|_{C^{0,\alpha}_{-2,\delta;t_i}} \\
&\leq \left\|\eta_{t_i}L_t\underline{a}\right\|_{C^{0,\alpha}_{-2,\delta;t_i}} + \left\|\eta_{t_i}L_{t_i}\overline{\pi}_{t_i}\underline{a}_i\right\|_{C^{0,\alpha}_{-2,\delta;t_i}} + \left\|\overline{\pi}_{t_i}L_{t_i}\eta_{t_i}\underline{a}_i\right\|_{C^{0,\alpha}_{-2,\delta;t_i}} \\
&\leq \left\|\eta_{t_i}L_t\underline{a}\right\|_{C^{0,\alpha}_{-2,\delta;t_i}} + ||1||_{C^{0,\alpha}_{0,\delta;t_i}} \left\|\eta_{t_i}L_{t_i}\overline{\pi}_{t_i}\underline{a}_i\right\|_{C^{0,\alpha}_{-2,0;t_i}} + t^{1-\alpha}\left\|\pi_{t_i}L_{t_i}\eta_{t_i}\underline{a}_i\right\|_{C^{0,\alpha}} \\
&\leq c\left(\left\|\eta_{t_i}L_t\underline{a}\right\|_{C^{0,\alpha}_{-2,\delta;t_i}} + ct^{\delta/2}t^{2-\alpha}\left\|\pi_t\underline{a}_i\right\|_{C^{1,\alpha}} + t^{1-2\alpha}\left\|\eta_{t_i}\underline{a}_i\right\|_{C^{1,\alpha}_{-1,0;t}}\right) \\
&\leq c\left(\left\|\eta_{t_i}L_t\underline{a}\right\|_{C^{0,\alpha}_{-2,\delta;t_i}} + O(t^{\delta/2+1-\alpha}) + O(t^{1-2\alpha+\delta/2})\right)
\end{aligned}
$$

where we used Proposition 4.105 in the first step; we used $\overline{\pi}_{t_i} + \eta_{t_i} = 1$ in the second and third steps; Propositions 4.21 and 4.65 in the fourth step; and Proposition 4.122 together with $||1||_{C^{0,\alpha}_{0,\delta;t_i}} \leq ct^{\delta/2}$ in the fifth step. Multiplying the last line with $t_i^{-\delta/2}$, the last two summands tend to zero as they are bounded by positive powers of $t$. The first summand tends to zero by the assumption $\left\|L_t\underline{a}_i\right\|_{\mathfrak{Y}_t} \to 0$.

It remains to prove that

$$
t_i\left\|\pi_{t_i}\underline{a}_i\right\|_{C^{1,\alpha}} \to 0. \tag{4.126}
$$

We have that

$$
\begin{aligned}
\lim_{i\to\infty} t_i\left\|\pi_{t_i}\underline{a}_i\right\|_{C^{1,\alpha}} &\leq \lim_{i\to\infty} t_i\left\|\pi_{t_i}L_{t_i}\iota_{t_i}\pi_{t_i}\underline{a}_i\right\|_{C^{0,\alpha}} \\
&\leq \lim_{i\to\infty} t_i\left(\left\|\pi_tL_t\underline{a}\right\|_{C^{0,\alpha}} + \left\|\pi_tL_t\eta_t\underline{a}\right\|_{C^{0,\alpha}}\right) \\
&\leq \lim_{i\to\infty} t_i\left(\left\|\pi_tL_t\underline{a}\right\|_{C^{0,\alpha}} + ct^{-\alpha}\left\|\eta_t\underline{a}\right\|_{C^{1,\alpha}_{-1,0;t}}\right).
\end{aligned}
$$

where we used Proposition 4.81 in the first step, $\overline{\pi}_{t_i} + \eta_{t_i} = 1$ in the second step, Proposition 4.122 in the third step. Here, the second summand tends to zero by Eq. (4.125), and the first summand tends to zero by the assumption $\left\|L_t\underline{a}_i\right\|_{\mathfrak{Y}_t} \to 0$. Altogether, $\left\|\underline{a}_i\right\|_{\mathfrak{X}_t} \to 0$, which is a contradiction. □

## 4.4 Quadratic Estimate

We state an estimate for the quadratic form $Q_t$ from Eq. (4.61), where we denote its associated bilinear form by the same symbol. This statement is taken from [Wal17] and the proof can be directly adapted to our slightly different setting.

**Proposition 4.127** (Proposition 9.1 in [Wal17]). *There exists a constant $c > 0$ such that for $t \in (0,1)$ we have*

$$\begin{aligned} &\left\|\eta_t Q_t(\underline{a}_1, \underline{a}_2)\right\|_{C^{0,\alpha}_{-2,\delta;t}} \\ &\quad \leq ct^{-\alpha}\Big(\left\|\eta_t \underline{a}_1\right\|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left\|\eta_t \underline{a}_2\right\|_{C^{0,\alpha}_{-1,\delta;t}} + \left\|\eta_t \underline{a}_1\right\|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left\|\pi_t \underline{a}_2\right\|_{C^{0,\alpha}} \\ &\qquad + \left\|\pi_t \underline{a}_1\right\|_{C^{0,\alpha}} \cdot \left\|\eta_t \underline{a}_2\right\|_{C^{0,\alpha}_{-1,\delta;t}} + \left\|\pi_t \underline{a}_1\right\|_{C^{0,\alpha}} \cdot \left\|\pi_t \underline{a}_2\right\|_{C^{0,\alpha}}\Big) \end{aligned} \tag{4.128}$$

*and*

$$\begin{aligned} &t\left\|\pi_t Q_t(\underline{a}_1, \underline{a}_2)\right\|_{C^{0,\alpha}} \\ &\quad \leq ct^{-\alpha}\Big(\left\|\eta_t \underline{a}_1\right\|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left\|\eta_t \underline{a}_2\right\|_{C^{0,\alpha}_{-1,\delta;t}} + \left\|\eta_t \underline{a}_1\right\|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left\|\pi_t \underline{a}_2\right\|_{C^{0,\alpha}} \\ &\qquad + \left\|\pi_t \underline{a}_1\right\|_{C^{0,\alpha}} \cdot \left\|\eta_t \underline{a}_2\right\|_{C^{0,\alpha}_{-1,\delta;t}} + t\left\|\pi_t \underline{a}_1\right\|_{C^{0,\alpha}} \cdot \left\|\pi_t \underline{a}_2\right\|_{C^{0,\alpha}}\Big). \end{aligned} \tag{4.129}$$

## 4.5 Deforming to Genuine Solutions

In this subsection we will complete the construction of $G_2$-instantons and show that in two favourable situations the $G_2$-instanton $\theta$ and the Fueter section $s$ can be glued together to a $G_2$-instanton on $N_t$. The favourable situations are:

1. The Fueter section is a section of rigid ASD-instantons (cf. Theorem 4.130). This implies, in particular, that the Fueter section is infinitesimally rigid. In this case the map $\pi_t$ from Definition 4.62 is just the zero map, which leads to better estimates of the quadratic part $Q_t$ of the instanton equation.
2. We are in the special situation of Eq. (4.58), where we resolved the orbifold $T^7/\Gamma$.

The main reason we are confined to these two favourable scenarios is the following: in Corollaries 4.54 and 4.57 we proved a pregluing estimate with a good power of $t^{1/18}$ in the general case and a good power of $t^2$ in the case of $T^7/\Gamma$, roughly speaking. In Proposition 4.127 we stated an estimate for the quadratic part of the instanton operator which in particular implies

$$\left\|Q_t(\underline{a}_1, \underline{a}_2)\right\|_{\mathfrak{Y}} \le t^{-2-\alpha-\delta/2} \left\|\underline{a}_1\right\|_{\mathfrak{X}} \left\|\underline{a}_2\right\|_{\mathfrak{X}}.$$

To apply the inverse function theorem, we would need the bad power $t^{-2-\alpha-\delta/2}$ from this estimate to be absorbed by the good power from the pregluing estimate, but the pregluing estimate is only good enough to do this in the case of the orbifold $T^7/\Gamma$. If the Fueter section is actually the constant section of a rigid ASD-instanton, then we have a better estimate for the quadratic part of the instanton equation.

**Theorem 4.130.** *Assume now that the section $s$ is given by a rigid ASD-instanton in every point $x \in L$, and assume that the connection $\theta$ used to define the approximate $G_2$-instanton $A_t$ from Proposition 4.27 is infinitesimally rigid.*

*There exists $c > 0$ such that for small $t$ there exists $\underline{a}_t = (a_t, \xi_t) \in C^{1,\alpha}(\Omega^0 \oplus \Omega^1(\operatorname{Ad} E_t))$ such that $\widetilde{A}_t := A_t + a_t$ is a $G_2$-instanton. Furthermore, $\underline{a}_t$ satisfies $\left\|\underline{a}_t\right\|_{C^{1,\alpha}_{-1,\delta;t}} \le ct^{1/18}$.*

**Theorem 4.131.** *Let $N \to Y'$ be the resolution of the orbifold $Y' = T^7/\Gamma$ from before. Assume that the connection $\theta$ used to define the approximate $G_2$-instanton $A_t$ from Proposition 4.27 is infinitesimally rigid and that $s$ is an infinitesimally rigid Fueter section.*

*There exists $c > 0$ such that for small $t$ there exists an $\underline{a}_t = (a_t, \xi_t) \in C^{1,\alpha}(\Omega^0 \oplus \Omega^1(\operatorname{Ad} E_t))$ such that $\widetilde{A}_t := A_t + a_t$ is a $G_2$-instanton. Furthermore, $\underline{a}_t$ satisfies $\left\|\underline{a}_t\right\|_{\mathfrak{X}_t} \le ct^{2-2\alpha}$.*

The proof of the theorems will use the following lemma:

**Lemma 4.132** (Lemma 7.2.23 in [DK90]). *Let $X$ be a Banach space and let $T : X \to X$ be a smooth map with $T(0) = 0$. Suppose there is a constant $c > 0$ such that*

$$||Tx - Ty|| \le c(||x|| + ||y||)\,||x - y||.$$

*Then if $y \in X$ satisfies $||y|| \leq \frac{1}{10c}$, there exists a unique $x \in X$ with $||x|| \leq \frac{1}{5c}$ solving*

$$x + Tx = y.$$

*The unique solution satisfies $||x|| \leq 2\,||y||$.*

*Proof of Theorem 4.130.* In the case that $s$ is a section of rigid ASD instantons, we have that the projection map $\pi_t$ is zero. Therefore, Propositions 4.103 and 4.105 give

$$\left\|\underline{a}\right\|_{C^{1,\alpha}_{-1,\delta;t}} \leq c\left\|L_t\underline{a}\right\|_{C^{0,\alpha}_{-2,\delta;t}}. \tag{4.133}$$

This means that

$$L_t : C^{1,\alpha}((\Omega^0 \oplus \Omega^1)(N_t, \operatorname{Ad} E_t)) \to C^{1,\alpha}((\Omega^0 \oplus \Omega^1)(N_t, \operatorname{Ad} E_t))$$

is injective. Because $L_t$ is formally self-adjoint, it is also bijective. Denote its inverse by $L_t^{-1}$. Furthermore, using $\pi_t = 0$, and therefore $\eta_t = \mathrm{Id}$, Proposition 4.127 gives

$$\left\|Q_t(\underline{a}_1, \underline{a}_2)\right\|_{C^{0,\alpha}_{-2,\delta;t}} \leq ct^{-\alpha}\left\|\underline{a}_1\right\|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left\|\underline{a}_2\right\|_{C^{0,\alpha}_{-1,\delta;t}}. \tag{4.134}$$

Set $T_t := Q_t \circ L_t^{-1}$. We then have

$$\begin{aligned}
\left\|T_t(\underline{b}_1) - T_t(\underline{b}_2)\right\|_{C^{0,\alpha}_{-2,\delta;t}} &= \left\|Q(L^{-1}\underline{b}_1 - L^{-1}\underline{b}_2, L^{-1}\underline{b}_1 + L^{-1}\underline{b}_2)\right\|_{C^{0,\alpha}_{-2,\delta;t}} \\
&\leq ct^{-\alpha}\left\|L^{-1}\underline{b}_1 - L^{-1}\underline{b}_2\right\|_{C^{0,\alpha}_{-1,\delta;t}}\left\|L^{-1}\underline{b}_1 + L^{-1}\underline{b}_2\right\|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\leq ct^{-\alpha}\left\|L^{-1}\underline{b}_1 - L^{-1}\underline{b}_2\right\|_{C^{1,\alpha}_{-1,\delta;t}}\left\|L^{-1}\underline{b}_1 + L^{-1}\underline{b}_2\right\|_{C^{1,\alpha}_{-1,\delta;t}} \\
&\leq ct^{-\alpha}\left\|\underline{b}_1 - \underline{b}_2\right\|_{C^{0,\alpha}_{-2,\delta;t}}\left(\left\|\underline{b}_1\right\|_{C^{0,\alpha}_{-2,\delta;t}} + \left\|\underline{b}_1\right\|_{C^{0,\alpha}_{-2,\delta;t}}\right),
\end{aligned}$$

where we used Eq. (4.134) in the first inequality and Eq. (4.133) in the last inequality.

For $e_t$ we have

$$||e_t||_{C^{0,\alpha}_{-2,0;t}} \leq ct^{1/18}$$

by Corollary 4.54. For small $t$, we have that $t^{1/18} < \left(t^{-\alpha+\delta/2}\right)^{-1}$ due to our choices of $\alpha$ and $\delta$

in Definition 4.19. Thus, by applying Lemma 4.132 to the map $T_t$, we get a solution $\underline{b}_t$ to the equation $\underline{b}_t + T_t(\underline{b}_t) = -e_t$ for small $t$, satisfying the estimate $\left\|\underline{b}_t\right\|_{C^{0,\alpha}_{-2,0;t}} \leq ct^{1/18}$.

Letting $\underline{a}_t := L_t^{-1}(\underline{b}_t)$, this means precisely $L_t(\underline{a}_t) + Q_t(\underline{a}_t) = -e_t$, so $\widetilde{A}_t = A_t + a_t$ is a $G_2$-instanton, and $\underline{a}_t$ satisfies $\left\|\underline{a}_t\right\|_{C^{1,\alpha}_{-1,\delta;t}} \leq ct^{1/18}$ by Eq. (4.133), which proves the claim. □

*Proof of Theorem 4.131.* As in the proof of Theorem 4.130, set $T_t := Q_t \circ L_t^{-1}$. Then

$$
\begin{aligned}
&\left\|T_t(\underline{b}_1) - T_t(\underline{b}_2)\right\|_{\mathfrak{Y}_t} \\
&= \left\|Q(L^{-1}\underline{b}_1 - L^{-1}\underline{b}_2, L^{-1}\underline{b}_1 + L^{-1}\underline{b}_2)\right\|_{\mathfrak{Y}_t} \\
&= t^{-\delta/2}\left\|\eta_t Q(L^{-1}\underline{b}_1 - L^{-1}\underline{b}_2, L^{-1}\underline{b}_1 + L^{-1}\underline{b}_2)\right\|_{C^{0,\alpha}_{-2,\delta;t}} \\
&\qquad + t\left\|\pi_t Q(L^{-1}\underline{b}_1 - L^{-1}\underline{b}_2, L^{-1}\underline{b}_1 + L^{-1}\underline{b}_2)\right\|_{C^{0,\alpha}} \\
&\leq ct^{-\alpha-\delta/2}\Big(\left\|\eta_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right\|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left\|\eta_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right\|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\qquad\qquad + \left\|\eta_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right\|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left\|\pi_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right\|_{C^{0,\alpha}} \\
&\qquad\qquad + \left\|\pi_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right\|_{C^{0,\alpha}} \cdot \left\|\eta_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right\|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\qquad\qquad + \left\|\pi_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right\|_{C^{0,\alpha}} \cdot \left\|\pi_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right\|_{C^{0,\alpha}}\Big) \\
&\qquad + ct^{-\alpha}\Big(\left\|\eta_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right\|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left\|\eta_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right\|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\qquad\qquad + \left\|\eta_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right\|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left\|\pi_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right\|_{C^{0,\alpha}} \\
&\qquad\qquad + \left\|\pi_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right\|_{C^{0,\alpha}} \cdot \left\|\eta_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right\|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\qquad\qquad + t\left\|\pi_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right\|_{C^{0,\alpha}} \cdot \left\|\pi_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right\|_{C^{0,\alpha}}\Big) \\
&\leq ct^{-\alpha-2-\delta/2}\Big(t^{-\delta}\left\|\eta_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right\|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left\|\eta_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right\|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\qquad\qquad + t^{1-\delta/2}\left\|\eta_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right\|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left\|\pi_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right\|_{C^{0,\alpha}} \\
&\qquad\qquad + t^{1-\delta/2}\left\|\pi_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right\|_{C^{0,\alpha}} \cdot \left\|\eta_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right\|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\qquad\qquad + t^2\left\|\pi_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right\|_{C^{0,\alpha}} \cdot \left\|\pi_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right\|_{C^{0,\alpha}}\Big) \\
&\leq ct^{-\alpha-2-\delta/2}\left\|L^{-1}(b_1 - b_2)\right\|_{\mathfrak{X}_t}\left\|L^{-1}(b_1 + b_2)\right\|_{\mathfrak{X}_t} \\
&\leq ct^{-\alpha-2-\delta/2}\,||b_1 - b_2||_{\mathfrak{Y}_t}\,||b_1 + b_2||_{\mathfrak{Y}_t} \\
&\leq ct^{-\alpha-2-\delta/2}\,||b_1 - b_2||_{\mathfrak{Y}_t}\left(||b_1||_{\mathfrak{Y}_t} + ||b_2||_{\mathfrak{Y}_t}\right).
\end{aligned}
$$

Here we used Proposition 4.127 in the third step, and Proposition 4.77 in the second to last step.

We have

$$||e_t||_{\mathfrak{Y}_t} \leq ct^{2-\alpha},$$

by Corollary 4.57. Applying Lemma 4.132 as in the proof of Theorem 4.130 shows the claim. □

## 4.6 An example Coming from a Stable Bundle

### 4.6.1 Review of the Resolution of $(T^3 \times \mathbf{K3})/\Gamma$

Recall the $G_2$-manifold constructed in [JK21, Section 7.3]: consider the sextic

$$C = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 : z_0^6 + z_1^6 + z_2^6 = 0\} \subset \mathbb{CP}^2$$

and let $\pi : X \to \mathbb{CP}^2$ be the double cover of $\mathbb{CP}^2$ branched over $C$. Then $X$ is a complex K3 surface with a Hyperkähler triple of Kähler forms $\omega^I, \omega^J, \omega^K$, cf. [Huy16, Example 1.3]. On $X$ we can define the following two maps: first, the map $\alpha : X \to X$ which swaps the two sheets of the branched cover. Second, there are two lifts $X \to X$ of the complex conjugation map $\sigma : \mathbb{CP}^2 \to \mathbb{CP}^2$. One of these two lifts acts freely on $X$, the other one does not. Denote the lift that does not act freely on $X$ by $\beta : X \to X$, which has $\mathrm{fix}(\beta) = \pi^{-1}(\mathbb{RP}^2) \simeq S^2$. The Hyperkähler triple $\omega^I, \omega^J, \omega^K$ can be chosen to satisfy

$$\begin{aligned}
\alpha^*\omega^I &= \omega^I, & \alpha^*\omega^J &= -\omega^J, & \alpha^*\omega^K &= -\omega^K,\\
\beta^*\omega^I &= -\omega^I, & \beta^*\omega^J &= \omega^J, & \beta^*\omega^K &= -\omega^K.
\end{aligned}$$

Let $\alpha, \beta$ act on $T^3$ via

$$\alpha(x_1, x_2, x_3) = (x_1, -x_2, -x_3), \beta(x_1, x_2, x_3) = \left(-x_1, x_2, \frac{1}{2} - x_3\right).$$

Denote $\Gamma = \langle \alpha, \beta \rangle$. Then $\alpha, \beta : T^3 \times X \to T^3 \times X$ preserve the product $G_2$-structure $\varphi$ on $T^3 \times X$ defined by equation Eq. (2.27). Furthermore, $\mathrm{fix}(\alpha) = 4 \cdot (S^1 \times C)$, $\mathrm{fix}(\beta) = 4 \cdot (S^1 \times S^2)$, where the $S^2$-factors are the double cover of $\mathrm{fix}(\sigma) = \mathbb{RP}^2 \subset \mathbb{CP}^2$. Therefore, $L = \mathrm{fix}(\alpha) \cup \mathrm{fix}(\beta)$ admits a nowhere vanishing harmonic 1-form, namely the parallel 1-form in the $S^1$-direction of each component. Thus, this orbifold is of the type considered in Section 3 and its resolution $N_t \to (T^3 \times X)/\Gamma$ admits a 1-parameter family of $G_2$-structures with small torsion, inducing metrics $g_t$, which can be perturbed to torsion-free $G_2$-structures inducing metrics $\widetilde{g}_t$.

### 4.6.2 A Connection on the Orbifold $(T^3 \times \mathrm{K3})/\Gamma$ coming from a Stable Bundle

We will now make use of the SO(3)-bundle $F$ over $\mathbb{CP}^2$ from Section 2.5.2. To this end, we first recall its definition. The tangent bundle $E$ of $\mathbb{CP}^2$ is a complex vector bundle of rank 2, which induces an SO(3)-bundle $F$ by Proposition 2.90. The Levi-Civita connection on $E$ is a Hermite-Einstein connection by Proposition 2.85 and induces an ASD instanton on $F$ by Proposition 2.90, denoted by $A$. We denote the standard Kähler structure on $\mathbb{CP}^2$ by $(J, g = g_{\mathrm{FS}}, \omega)$, where $g_{\mathrm{FS}}$ is the Fubini-Study metric. The pullback $\pi^* A$ is then an ASD instanton on the bundle $\pi^* F$ over $(X, \pi^* g)$, but it need not be ASD with respect to the Calabi-Yau metric on $X$. We will show in Corollary 4.136 that $\pi^* F$ also carries an instanton with respect to the Calabi-Yau metric.

**Proposition 4.135** (Lemma 9.1.9 in [DK90]). *The bundle $\pi^* E$ is stable with respect to $\omega$.*

**Corollary 4.136.** *The bundle $\pi^* E$ is stable with respect to the Calabi-Yau Kähler form $\omega^I$.*

*Proof of Corollary 4.136.* Denote by $\hat{\omega} = \pi^* \omega$ the pullback of the Kähler form for the Fubini-Study metric on $\mathbb{CP}^2$ to $X$. By Yau's proof of the Calabi conjecture we have that $\omega^I = \hat{\omega} + i\partial\bar{\partial}\phi$ for some $\phi : X \to \mathbb{R}$. In particular, $\omega^I$ and $\hat{\omega}$ are in the same de Rham cohomology class.

By Proposition 4.135, $\pi^* E$ is stable with respect to $\omega$. The Kähler form enters into the definition of stability only through the definition of slope. But slopes do not change when switching between $\omega^I$ and $\hat{\omega}$ as they are in the same cohomology class. Thus $\pi^* E$ is also stable with respect to $\omega^I$. $\square$

We also have the following:

**Corollary 4.137** (p. 348 in [DK90]). *Denote by $\pi_F : F \longrightarrow \mathbb{CP}^2$ the* $\mathrm{SO}(3)$*-bundle over* $\mathbb{CP}^2$ *from Section 2.5.2. Let $\pi : X \longrightarrow \mathbb{CP}^2$ be the branched double cover from Section 4.6.1 with Calabi-Yau metric $\hat{g}$. Then the bundle*

$$\hat{F} = \pi^* F = \{(x, u) \in X \times F : \pi_F(u) = \pi(x)\} \tag{4.138}$$

*admits an infinitesimally rigid and unobstructed ASD instanton $\hat{A}$ with respect to $\hat{g}$.*

*Proof.* The bundle $\pi^* E$ is stable with respect to $\omega^I$, and therefore admits a unique Hermite-Einstein connection by Theorem 2.83. Thus, we get an $\mathrm{SO}(3)$-bundle $\hat{F}$ with ASD instanton $\hat{A}$ by Proposition 2.90. Unobstructedness and infinitesimal rigidity of $\hat{A}$ are proved in [DK90, p. 348]. □

Pulling back $(\hat{F}, \hat{A})$ under the projection onto the second factor, $p : T^3 \times X \longrightarrow X$, gives a bundle with $G_2$-instanton by Example 2.98. Denote the bundle by $E_0$ and the connection by $\theta$. The connection $\hat{A}$ was infinitesimally rigid, and the following proposition implies that $\theta$ is infinitesimally rigid:

**Proposition 4.139.** *Let $I$ be an ASD instanton on a bundle $P$ over a compact 4-fold $Y$ with deformation operator $\delta_I$. Let $p : T^3 \times Y \longrightarrow Y$ be the projection onto the second factor. Then the $G_2$-instanton $p^* I$ is infinitesimally rigid if and only if $I$ is infinitesimally rigid and unobstructed.*

*Proof.* The pulled back connection $p^* I$ is a $G_2$-instanton by Example 2.98.

We first prove that $p^* I$ is infinitesimally rigid if $I$ is infinitesimally rigid and unobstructed. We will use Lemma 3.38 to derive an analog of Proposition 4.85 in this compact setting:

Suppose $\underline{a} \in (\Omega^0 \oplus \Omega^1)(T^3 \times Y, \mathrm{Ad}\, p^* P)$ satisfies $L_{p^* I} \underline{a} = 0$. Then $0 = L^*_{p^* I} L_{p^* I} \underline{a} = (\Delta_{\mathbb{R}^3} + D_I^* D_I) \underline{a}$ by Eq. (4.84), where $D_I^* D_I$ is an elliptic operator of second order. Because $Y$ is compact, it has bounded geometry, and $D_I^* D_I$ is *uniformly* elliptic and its coefficients and their first derivatives are uniformly bounded. So, by Lemma 3.38, $\underline{a}$ is independent of the $T^3$-direction. By Proposition 4.83, $\underline{a}$ is the pullback of an element in $\operatorname{Ker} \delta_I$ or the pullback of an element in $\operatorname{Ker}\delta_I^*$. By assumption, $I$ is infinitesimally rigid (i.e. $\operatorname{Ker}\delta_I = 0$) and unobstructed (i.e. $\operatorname{Ker}\delta_I^* = 0$), which proves the claim.

The converse direction follows directly from Proposition 4.83. $\square$

The gluing theorems Theorems 4.130 and 4.131 require a connection on the orbifold, $(T^3 \times X)/\Gamma$. The following proposition states that $\theta$ can be viewed as such a connection:

**Proposition 4.140**. *There exist lifts $\alpha_0 : E_0 \to E_0$ of $\alpha$ and $\beta_0 : E_0 \to E_0$ of $\beta$ such that $\alpha_0^2 = \beta_0^2 = \mathrm{Id}$, $\alpha_0^*\theta = \beta_0^*\theta = \theta$, $\alpha_0$ being the identity over $\mathrm{fix}(\alpha)$, and $\beta_0$* not *being the identity over $\mathrm{fix}(\beta)$.*

This relies on the following construction on $X$:

**Proposition 4.141**. *There exists a lift $\hat{\beta} : \hat{F} \to \hat{F}$ of $\beta$ such that $\hat{\beta}^2 = \mathrm{Id}$, $\hat{\beta}^*\hat{A} = \hat{A}$, and $\hat{\beta}$ not being the identity over $\mathrm{fix}(\beta)$.*

*Proof.* Denote by $\sigma : \mathbb{CP}^2 \to \mathbb{CP}^2$ the conjugation map and $E = T\mathbb{CP}^2$ as before. We can then view $\mathrm{d}\sigma$ as a complex linear map $E \to \overline{E}$ covering $\sigma$. Define

$$\begin{aligned} \hat{\sigma} : E \otimes \overline{E} &\to E \otimes \overline{E} \\ v \otimes w &\mapsto -\mathrm{d}\sigma w \otimes \mathrm{d}\sigma v, \end{aligned} \tag{4.142}$$

which is a complex linear map covering $\sigma : \mathbb{CP}^2 \to \mathbb{CP}^2$.

The manifold $\mathbb{CP}^2$ is Kähler, so the Levi-Civita connection $\nabla^{\mathrm{LC}}$ on $E$ is a Hermite-Einstein connection. The connection $\nabla^{\mathrm{LC}}$ on $E$ induces the product connection $\nabla^{\otimes}$ on $E \otimes \overline{E}$, which is again a Hermite-Einstein connection. We have that $\sigma$ is an isometry, so $\nabla^{\otimes}$ is preserved by $\hat{\sigma}$ in the sense of $\hat{\sigma} \circ \sigma^*\nabla^{\otimes} \circ \hat{\sigma} = \nabla^{\otimes}$.

Let $\hat{\beta}$ be the lift of $\hat{\sigma}$ to $\pi^*E \otimes \overline{\pi^*E}$, i.e. $\hat{\beta} : \pi^*E \otimes \overline{\pi^*E} \to \pi^*E \otimes \overline{\pi^*E}$ covering $\beta : X \to X$ and satisfying $p\hat{\beta} = \hat{\sigma}p$, where $p : \pi^*E \otimes \overline{\pi^*E} \to E \otimes \overline{E}$ is the obvious projection map. Then $\hat{\sigma}^*\nabla^{\otimes} = \nabla^{\otimes}$ implies $\hat{\beta}^*(\pi^*\nabla^{\otimes}) = \pi^*\nabla^{\otimes}$.

If $p \in \mathbb{CP}^2$ and $(u_1, u_2)$ is a unitary basis of $E_p$, then $(\mathrm{d}\sigma(u_1), \mathrm{d}\sigma(u_2))$ is a unitary basis of $E_{\sigma(p)}$, and writing elements of the trace-free unitary endomorphism bundle $\mathfrak{u}_0(\pi^*E)$ in these bases, we see that $\hat{\beta}$ acts as

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \mapsto -\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$
$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mapsto -\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Thus, $\hat{\beta}$ induces a map on $\hat{F} = \mathrm{SO}(\mathfrak{u}_0(\pi^*E))$ that is not the identity over fix$(\beta)$ and preserves the ASD connection $\hat{A}$ on $\hat{F}$ induced by $\pi^*\nabla^{\otimes}$ according to Proposition 2.90. □

*Remark* 4.143. This only works because we have a lift of complex conjugation $\sigma : \mathbb{CP}^2 \to \mathbb{CP}^2$ to $F$ in Proposition 4.141. It follows from Proposition 2.92 that no lift of $\sigma$ to $E$ exists, so it is important to change from $\mathrm{U}(2)$-bundles to $\mathrm{SO}(3)$-bundles in this example.

*Remark* 4.144. Without the minus sign in Eq. (4.142), $\hat{\beta}$ would not descend to a map on $\mathrm{SO}(\mathfrak{u}_0(\pi^*E))$. That is because the map $-\mathrm{Id} : \mathfrak{u}_0(\pi^*E) \to \mathfrak{u}_0(\pi^*E)$ is orientation reversing, because $\mathfrak{u}_0(\pi^*E)$ has odd rank.

*Proof of Proposition 4.140.* The bundle $\hat{F}$ from Eq. (4.138) is the pullback of a bundle $F$ from $\mathbb{CP}^2$ to $X$, thus we have the natural map

$$\begin{aligned} \hat{\alpha} : \hat{F} &\to \hat{F} \\ (x, u) &\mapsto (\alpha(x), u) \end{aligned}$$

covering $\alpha : X \to X$. The bundle $E_0$ is the pullback of $\hat{F}$ to $T^3 \times X$, and we can canonically extend the map $\hat{\alpha}$ and the map $\hat{\beta}$ from Proposition 4.141 to $E_0$ and find that they have the required properties. □

Because of Proposition 4.140, the connection $\theta$ defines a connection on the orbifold $(T^3 \times \mathrm{K3})/\Gamma$. The holonomy of $\theta$ around the four $S^1 \times C \subset (T^3 \times X)/\Gamma$ fixed by $\alpha$ is trivial, and the holonomy around the four $S^1 \times S^2$ fixed by $\beta$ has order 2.

### 4.6.3 The Resulting Connection on the Resolution of $(T^3 \times \mathrm{K3})/\Gamma$

**Corollary 4.145.** *For small $t$, there exists an irreducible $G_2$-instanton with structure group* $\mathrm{SO}(3)$ *on the resolution $N_t$ of $(T^3 \times X)/\Gamma$.*

*Proof.* We make use of the $\alpha$-invariant and $\beta$-invariant connection $\theta$ from Proposition 4.140 over $(T^3 \times X)/\Gamma$.

Next consider the product connection $A_0$ on the trivial $\mathrm{SO}(3)$-bundle over Eguchi-Hanson space $X_{\mathrm{EH}}$. The holonomy representation at infinity of the product connection is trivial, i.e. $\rho_0 : \Gamma \to \mathrm{SO}(3)$, $\rho_0(\pm 1) = \mathrm{Id}$, thus $G_{\rho_0} = G$, where $G_{\rho_0}$ was defined in Eq. (2.43). $A_0$ is infinitesimally rigid, which can for example be seen from the dimension formula in Theorem 2.52, so for each copy of $S^1 \times C \subset (T^3 \times X)/\Gamma$ fixed by $\alpha$ we have that

$$\begin{aligned} S^1 \times C &\to \mathrm{Fr} \times E_0|_{S^1 \times C} \times_{\mathrm{U}(2) \times G} M \\ x &\mapsto [(f,u), [A_0]] \text{ for } f \in \mathrm{Fr}_x,\, u \in (E_0)_x \text{ arbitrary} \end{aligned}$$

is a well-defined map, parallel, and therefore a Fueter section.

Likewise, let $A_{0,1}$ be the ASD instanton over $X_{\mathrm{EH}}$ from Proposition 2.54. This is defined on a $\mathrm{U}(1)$-bundle and we view it as a reducible $\mathrm{SO}(3)$-connection. This has non-trivial holonomy $\rho_{0,1} : \Gamma \to \mathrm{SO}(3)$ at infinity, thus $G_{\rho_{0,1}} \subsetneq G$. For each copy of $S^1 \times S^2$ fixed by $\beta$ we find that

$$\begin{aligned} S^1 \times S^2 &\to \mathrm{Fr} \times E_0|_{S^1 \times S^2} \times_{\mathrm{U}(2) \times G_{\rho_{0,1}}} M \\ x &\mapsto [(f,u), [A_{0,1}]] \text{ for } f \in \mathrm{Fr}_x,\, u \in (E_0)_x \text{ arbitrary} \end{aligned}$$

is a Fueter section. By Proposition 4.140, the connection $\theta$ and the eight Fueter sections satisfy the necessary compatibility condition from Proposition 4.27. Thus, Theorem 4.130 applies and gives a $G_2$-instanton $\widetilde{A}_t$ on $N_t$. The connections $\widetilde{A}_t$ converge to $\theta$ on compact subsets of $(T^3 \times X)/\Gamma \setminus \mathrm{fix}(\Gamma)$ as $t \to 0$. The connection $\theta$ has full holonomy $\mathrm{SO}(3)$, as otherwise the Fubini-Study metric on $\mathbb{CP}^2$ would need to have reduced holonomy. Thus, $\widetilde{A}_t$ has full holonomy for small $t$ and is therefore irreducible. □

# A Appendix

## A.1 The Isometry Group of Eguchi-Hanson Space

In Proposition 2.5 we defined the Eguchi-Hanson space $X_{\mathrm{EH}}$ and proved that it admits a Hyperkähler metric $g_{(k)}$. The following statement about the isometry group of $X_{\mathrm{EH}}$ is a standard fact, but we could not locate a proof of it in the literature, so we provide it here:

**Proposition A.1.** *For any $k > 0$,*

1. *the isometry group of the metric $g_{(k)}$ on $X_{EH}$ is isomorphic to* $\mathrm{SO}(3) \times \mathrm{O}(2)$*,*
2. *the group of isometries preserving the complex structure induced by $\omega_1^{(k)}$ is isomorphic to* $\mathrm{U}(2)/\{\pm 1\}$*,*
3. *the group of isometries preserving the three complex structures induced by $\omega_1^{(k)}$, $\omega_2^{(k)}$, and $\omega_3^{(k)}$ respectively is isomorphic to* $\mathrm{SO}(3)$*.*

*Proof.* The space $X_{\mathrm{EH}}$ contains $\mathrm{SO}(3) \times_{SO(2)} \{0\}$ as a unique minimal surface which must be mapped to itself by an isometry. Thus, an isometry must preserve the distance to this minimal surface, i.e. preserve the $\mathbb{R}_{\geq 0}$-factor of $\mathbb{R}_{\geq 0} \times_{\mathrm{SO}(2)} \mathrm{SO}(3)$. It thus suffices to find the isometry group of $\mathrm{SO}(3)$ endowed with the metric $(e^1(r))^2 + (e^2(r))^2 + (e^3(r))^2$ for some $r > 0$, where $e^1, e^2, e^3$ were defined in Proposition 2.5. As $r \to \infty$, this metric converges towards the metric induced by the round metric on $S^3$. Through this, an isometry of $X_{\mathrm{EH}}$ induces an isometry of $\mathbb{R}^4/\{\pm 1\}$, which has isometry group $\mathrm{SO}(4)/\{\pm 1\} \simeq \mathrm{SO}(3) \times \mathrm{SO}(3)$. This shows that $\mathrm{Isom}(X_{\mathrm{EH}}, g_{(k)}) \subset \mathrm{SO}(3) \times \mathrm{SO}(3)$, where the first $\mathrm{SO}(3)$ acts by left multiplication, and the second acts by right multiplication on the $\mathrm{SO}(3)$-factor of $X_{\mathrm{EH}}$.

A calculation in coordinates shows $\mathrm{Isom}(X_{\mathrm{EH}}, g_{(k)}) = \mathrm{SO}(3) \times \mathrm{O}(2) \hookrightarrow \mathrm{SO}(3) \times \mathrm{SO}(3)$, where

$$\begin{aligned} \mathrm{O}(2) &\hookrightarrow \mathrm{SO}(3) \\ A &\mapsto \begin{pmatrix} \det A & 0 \\ 0 & A \end{pmatrix}. \end{aligned}$$

Regarding the second point, a computation shows that the subgroup of isometries preserving the complex structure induced by $\omega_1^{(k)}$ and is exactly $\mathrm{SO}(3) \times \mathrm{SO}(2)$. This is isomorphic to $\mathrm{U}(2)/\{\pm 1\}$, which is seen from the split short exact sequence

$$1 \to \mathrm{SU}(2)/\{\pm 1\} \simeq \mathrm{SO}(3) \to \mathrm{U}(2)/\{\pm 1\} \stackrel{\det}{\to} \mathrm{U}(1) \simeq \mathrm{SO}(2) \to 1. \tag{A.2}$$

The last point is again a computation in coordinates. □

## A.2 Measuring Vectors in Nearby Metrics

In Section 3 we define several different metrics on a manifold, for example $g_t^P$, $g_t^N$, and $\widetilde{g}_t^N$. These metrics are all near to each other, in a suitable sense. In Section 4 we sometimes switch between these metrics: an estimate with respect to $g_t^P$ gives rise to an estimate with respect to $g_t^N$, provided the two metrics are near enough to each other. To be precise, we use the following result:

**Proposition A.3.** *Let $V$ be a vector space and let $g$ and $\widetilde{g}$ be inner products on $V$.*

1. *Let $v \in V$ such that $|v|_g \leq \delta$ and $|\widetilde{g} - g|_g \leq \epsilon$, then $|v|_{\widetilde{g}} \leq \delta + \delta\epsilon$.*
2. *Let $\omega \in V^*$ such that $|\omega|_g \leq \delta$ and $|\widetilde{g} - g|_{\widetilde{g}} \leq \epsilon$, then $|\omega|_{\widetilde{g}} \leq \delta + \delta\epsilon$.*

When integrating, we have the following estimate for switching from one volume form to another:

**Proposition A.4.** *Let $M$ be an oriented manifold, and $g$, $\widetilde{g}$, $h$ Riemannian metrics on $M$. Then*

$$\left| \int_M f \cdot \mathrm{vol}_g - \int_M f \cdot \mathrm{vol}_{\widetilde{g}} \right| \leq \int_M |f| \cdot |\, \mathrm{vol}_g - \mathrm{vol}_{\widetilde{g}}\,|_h \cdot \mathrm{vol}_h \tag{A.5}$$

*for all $f : M \to \mathbb{R}$ with the property that all the integrals in Eq.* (A.5) *are defined.*

## A.3 Rigidity of Finite Subgroups

Let $G$ be a compact connected Lie group and $\Gamma$ be a finite group. In Section 2.4.2 we took $\Gamma$ to be a finite subgroup of $\mathrm{SU}(2)$, thereby acting on $B^4$. An orbifold $G$-bundle over $B^4/\Gamma$ is a $G$-

bundle $P$ over $B^4$ together with a lift of the action of $\Gamma$ to $P$. In Eq. (2.43) we extended elements of $G$ to elements of the orbifold gauge group $\mathscr{G}(P)$. We could do this, because we assumed the lift of $\Gamma$ to act in a standard way on $P$, see Eq. (2.40) for the precise statement. In other words: we used that up to gauge equivalence, orbifold bundles over $B^4/\Gamma$ are determined by the homomorphism $\Gamma \to P_0 \simeq G$ induced by the lift of $\Gamma$ to $P$. The proof of this fact was given in Proposition 2.39, but used that the homomorphism $\Gamma \to G$ is rigid, in some sense. We make this rigidity precise here and prove that every finite group in a compact Lie group is rigid. The proof is taken from [Bad21], where also the generalisation to non-compact $G$ is explained.

*Definition* A.6. The set $\mathrm{Hom}(\Gamma, G) \subset G^{|\Gamma|}$ endowed with the restriction of the product topology on $G^{|\Gamma|}$ is called the *representation variety.*

*Definition* A.7. Let $E$ be a $\Gamma$-module. A map $b \in \Gamma \to E$ is called *cocycle* if

$$b(\gamma\delta) = b(\gamma) + \gamma \cdot b(\delta) \text{ for all } \gamma, \delta \in \Gamma.$$

We denote the set of cocycles by $Z^1(\Gamma, E)$. A map $b \in \Gamma \to E$ is called *coboundary* if there exists $v \in E$ such that

$$b(\gamma) = v - \gamma \cdot v \text{ for all } \gamma \in \Gamma.$$

We denote the set of coboundaries by $B^1(\Gamma, E) \subset Z^1(\Gamma, E)$. The *first cohomology of* $\Gamma$ *with coefficients in E is*

$$H^1(\Gamma, E) = Z^1(\Gamma, E)/B^1(\Gamma, E).$$

**Theorem A.8** (Point 3 in [Wei64]). *Fix a group homomorphism $r : \Gamma \to G$. The group $G$ is acting on $\mathfrak{g}$ through the adjoint representation, and together with $r$ this gives $\Gamma$ the structure of a $\Gamma$-module. If $H^1(\Gamma, \mathfrak{g}) = 0$, then there exists a neighbourhood $U \subset \mathrm{Hom}(\Gamma, G)$ of $r$ in which each element is conjugate to $r$, i.e. for all $s \in U$ there exists $g \in G$ such that*

$$s = l_g \circ r_{g^{-1}} \circ r.$$

*Here, $l_g, r_{g^{-1}} : G \to G$ denote left translation and right translation on $G$, respectively.*

*Definition* A.9. Fix $\pi : \Gamma \to \mathrm{Aut}(E)$. An *affine action* of $\Gamma$ on $E$ is a group homomorphism $\phi : \Gamma \to \mathrm{Aff}(E)$. We say that $\pi$ is the *linear part* of the affine action $\phi$ if for all $\gamma \in \Gamma$ there exists $v_0 \in E$ such that

$$\phi(\gamma)(v) = \pi(\gamma)(v) + v_0 \text{ for all } v \in E.$$

**Lemma A.10** (Lemma 2.1 in [DX16]). *The map* $\pi : \Gamma \longrightarrow \mathrm{Aut}(E)$ *endows* $\Gamma$ *with an* $E$*-module structure. We have* $H^1(\Gamma, E) = 0$ *with respect to this* $E$*-module structure if and only if every affine action with linear part* $\pi$ *has a fixed point.*

**Corollary A.11.** *The finite group* $\Gamma$ *with any* $E$*-module structure satisfies* $H^1(\Gamma, E) = 0$.

*Proof.* Let $\phi : \Gamma \longrightarrow \mathrm{Aff}(E)$ be an affine action. Then the element

$$X := \sum_{\delta \in \Gamma} \phi(\delta)(0) \in E$$

satisfies $\phi(\gamma)(X) = X$ for all $\gamma \in \Gamma$. By Lemma A.10 this implies that $H^1(\Gamma, E) = 0$. □

**Corollary A.12.** *The representation variety* $\mathrm{Hom}(\Gamma, G)$ *has finitely many connected components. For each connected component* $C$ *there exists* $r \in \mathrm{Hom}(\Gamma, G)$ *such that*

$$C = U_r := \{l_g \circ r_{g^{-1}} \circ r : g \in G\}.$$

*Proof.* Because $\Gamma$ is finite and $G$ is compact we have that $\mathrm{Hom}(\Gamma, G)$ is compact and therefore has finitely many connected components. Fix some $r \in \mathrm{Hom}(\Gamma, G)$. Then $U_r$ is compact because it is the image of $G$ under the conjugation map. Thus, $U_r$ is closed. On the other hand, $U_r$ is open by Theorem A.8 together with Corollary A.11. Thus, each connected component of $\mathrm{Hom}(\Gamma, G)$ is of the form $U_r$ for some $r \in \mathrm{Hom}(\Gamma, G)$. □

## A.4 Removable Singularities

In Definition 2.47 we defined a map from the moduli space of ASD connections over the Eguchi-Hanson space $X_{\mathrm{EH}}$ into the moduli space of ASD connections over the one point compactification of $X_{\mathrm{EH}}$. There, we used that every finite energy ASD connection that is defined over the complement of a point can be extended over this point. This statement was proved for Yang-Mills connections, not just ASD connections, in [Uhl82]. This is called the *Remov-*

*able Singularities Theorem.* Because our map between moduli spaces should be a map between *framed* moduli spaces, we need a version of the Removable Singularities Theorem that respects framings. This is Proposition A.14 and we then apply it to our special case of connections over $X_{\mathrm{EH}}$ in Corollary A.17.

**Theorem A.13** (Theorem 4.1 in [Uhl82], Theorem D.1 in [FU91]). *Let $G$ be a compact Lie group and $A$ be a connection on the trivial $G$-bundle over $B^4 \setminus \{0\}$, $A \in \mathscr{A}((B^4 \setminus \{0\}) \times G)$, which is in $L^2_{1,loc}$ and anti-self-dual with respect to a smooth metric on $B^4$. If*

$$\int_{B^4 \setminus \{0\}} |F(A)|^2 < \infty,$$

*then there exists an injective bundle homomorphism $\xi : (B^4 \setminus \{0\}) \times G \to B^4 \times G$ and a smooth connection $A' \in \mathscr{A}(B^4 \times G)$ such that $\xi^* A' = A$ over $B^4 \setminus \{0\}$.*

Theorem A.13 asserts existence of an extension over 0, and the following proposition asserts that this extension is essentially unique up to gauge:

**Proposition A.14.** *The data $\xi$ and $A'$ from Theorem A.13 are unique in the following sense: if $\xi', \xi'' : (B^4 \setminus \{0\}) \times G \to B^4 \times G$ and $A', A'' \in \mathscr{A}(B^4 \times G)$ are such that $(\xi')^* A' = (\xi'')^* A'' = A$, then the map $\xi'' \circ (\xi')^{-1} : (B^4 \setminus \{0\}) \times G \to (B^4 \setminus \{0\}) \times G$ can be extended to a continuous map $B^4 \times G \to B^4 \times G$.*

*Proof.* We view the connections $A', A''$ on the trivial bundle $B^4 \times G$ as elements in $\Omega^1(B^4, \mathfrak{g})$, and view the gauge transformation $\xi'' \circ (\xi')^{-1}$ as a map $B^4 \setminus \{0\} \to G$, denoted by $s$. Without loss of generality assume that $A'(0) = A''(0) = 0$, which can be arranged by composing $\xi', \xi''$ with a suitable gauge transformation of $B^4 \times G$. Then $A'' = s^* A'$ on $B^4 \setminus \{0\}$, thus

$$0 = A''(0) = \lim_{x \to 0} s^{-1}(x)\, \mathrm{d}s(x)$$

and by taking norms we see that $\lim_{x \to 0} \mathrm{d}s(x) = 0$. This implies that $\lim_{x \to 0} s(x)$ exists: if the limit does not exist, then we have two sequences $x_i, x_i' \to 0$ such that $\lim_{i \to \infty} s(x_i) \neq \lim_{i \to \infty} s(x_i')$. Without loss of generality assume that $x_i, x_i'$ can be joined by a line. The mean value theorem then gives a sequence $\theta_i \in B^4 \setminus \{0\}$ such that $|\,\mathrm{d}s(\theta_i)| \longrightarrow \infty$, which is a contradiction.

Therefore $\lim_{x\to 0} s(x)$ exists and defines a continuous map $\overline{s} : B^4 \longrightarrow G$, which in turn extends $\xi'' \circ (\xi')^{-1}$. □

Viewing the map $\xi$ from Theorem A.13 as a map $\xi : B^4 \setminus \{0\} \longrightarrow G$, the limit $\lim_{x\to 0} \xi(x)$ does not exist in general. But in important cases it does, according to the following proposition:

**Proposition A.15.** *Under the conditions of Theorem A.13, assume that $A$ is bounded, viewed as an element in $\Omega^1(B^4 \setminus \{0\}, \mathfrak{g})$. Viewing $\xi$ as a map $\xi : B^4 \setminus \{0\} \longrightarrow G$, we have that the limit*

$$\lim_{x\to 0} \xi(x) \in G$$

*exists.*

*Proof.* Without loss of generality assume that $A'(0) = 0$. Then,

$$\xi^* A'(x) = A(x) \text{ for all } x \in B^4 \setminus \{0\}. \tag{A.16}$$

Taking norms in Eq. (A.16) and using $\xi^* A'(x) = \xi^{-1}(x)\,\mathrm{d}\xi(x) + A'(x)$ we see that $\mathrm{d}\xi$ is bounded on $B^4 \setminus \{0\}$, and we can conclude the proof as in the proof of Proposition A.14. □

This can be applied to the case of ASD instantons on ALE manifolds:

**Corollary A.17.** *Let $P$ be a $G$-bundle over $X_{EH}$ and denote by $\mathscr{A}^{\mathrm{asd},-2}$ the set of ASD-connections on $P$ as in Eq. (2.43). Let $A_0 + a \in \mathscr{A}^{\mathrm{asd},-2}$, then there exists an orbifold $G$-bundle $P'$ over $\hat{X}_{EH}$ together with a connection $A' \in \mathscr{A}(P')$ and an injective bundle homomorphism $\xi : P \longrightarrow P'$ such that $\xi^* A' = A_0 + a$. Denote by $f : B^4/\Gamma \longrightarrow V$ the chart of $\hat{X}_{EH}$ around $\infty$ from Proposition 2.37. Fixing a trivialisation of $P$ over $V \setminus \{\infty\}$ induces a trivialisation of $P'$ over $V$ and we can view $\xi$ as a map $V \setminus \{\infty\} \longrightarrow G$. Then the limit $\lim_{x\to\infty} \xi(x)$, where $\infty \in \hat{X}_{EH}$, exists.*

*Proof.* The assumption $A_0 + a \in \mathscr{A}^{\mathrm{asd},-2}$ means that $a = O(r^{-2})$, measured in the ALE metric. By inspecting how the inversion $f$ acts on 1-forms, we find that $a = O(1)$, measured in the orbifold metric, and Proposition A.15 gives the claim. ◻

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