

Cohomogeneity one actions on symmetric spaces of noncompact type

First year mini-project

By: Daniel Platt daniel.platt.berlin@gmail.com Date: 29 June 2018 Advisor: Prof. Jürgen Berndt Abstract. Many desirable geometric properties of manifolds are governed by PDEs that are difficult to solve. Examples of these properties are Ricci-flatness, the Einstein condition, and holonomy properties. One approach to construct examples of spaces satisfying these properties is to assume that a Lie group G acts on the space such that the PDE is invariant under this action. This reduces the dimension of the problem, as a solution on a submanifold transversal to the orbits of the action is transported by the action of G along the orbits of the action.

The easiest case is that the orbits of the action have codimension 1 for most orbits. In this case, the group action is said to be of *cohomogeneity one* and the PDE is reduced to an ODE. This technique has been successfully applied to construct metrics with holonomy G_2 , nearly Kähler metrics on 6-manifolds, and Einstein metrics. It is therefore a natural problem to classify all cohomogeneity one actions.

In this work we lay out what is known about the classification of isometric G-actions on symmetric spaces of noncompact type. In the first section, we review the basic theory of symmetric spaces. In the second section, we present the necessary structure theory of real Lie algebras and in the following illustrate the different flavours of cohomogeneity one actions through examples. In particular, some observations are made about cohomogeneity one actions on the symmetric space $SO(3, 4)_0/(SO(3) SO(4))$ of rank 3.

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1 Symmetric Spaces

1.1 Symmetric Spaces as Riemannian Manifolds

Definition 1. Let (M, g) be a Riemannian manifold.

1. *M* is called *locally symmetric* if for all $x \in M$ there exists a neighborhood $U \subset M$ and an isometry $s: U \to U$ with

$$s(x) = x$$
, and $ds_x = -\operatorname{Id}$.

2. *M* is called *(globally) symmetric* if it is locally symmetric and at each point the isometry extends to an isometry $s: M \to M$.

Example 2. The manifold $S^n \subset \mathbb{R}^{n+1}$ with the standard round metric is globally symmetric. For $x \in S^n$, the symmetry $s : S^n \to S^n$ is given by reflection at the line $\mathbb{R}x$, i.e.

$$s(y) = -y + 2\langle x, y \rangle x. \tag{1}$$

The two following standard facts about symmetric spaces allow to check easily whether a given space is locally symmetric or not, and—in the case that the space is symmetric—immediately give the local symmetry:

Lemma 3 (Theorem 1.1 in [Hel01], Theorem 8.1.1 in [Wol11], Theorem 2.1 of [Tak91]). Let (M, g) be a Riemannian manifold.

- 1. (M,g) is locally symmetric if and only if the curvature tensor \mathcal{R} is parallel, i.e. $\nabla \mathcal{R} = 0$, where ∇ denotes the Levi-Civita-connection.
- 2. (M,g) is locally symmetric if and only if for all $x \in M$ the geodesic symmetry $\exp_x(X) \mapsto \exp_x(-X)$ is an isometry on a normal neighborhood of x.

1.2 Lie Theoretic Description of Symmetric Spaces

Definition 4. Let (M, g) be a symmetric space and $x \in M$ and let $G := I(M)_0$ be the identity component of the isometry group of M. The subgroup $K \subset G$ of isometries which fix x is called the *isotropy group in* x.

Denote by \mathfrak{k} be the Lie algebra of K, then vector fields $X \in \mathfrak{k}$ vanish in x.

Example 5. Figure 1 shows one example $X \in \mathfrak{k}$ for the isotropy group in the north pole x = (0, 0, 1). Note that X(0, 0, 1) = 0, which is the case for every element on \mathfrak{k} .

An element of the isotropy group K will move points on S^2 some distance along the arrows. That is, the isotropy group in x = (0, 0, 1) is exactly

$$K = \left\{ \begin{pmatrix} A \\ & 1 \end{pmatrix} \in \mathrm{SO}(3) : A \in \mathrm{SO}(2) \right\}.$$
 (2)



Figure 1: Example of an element $X \in \mathfrak{k}$

Theorem 6 (Theorem 3.3 in [Hel01], Theorem 8.3.4 in [Wol11], Theorem 2.3 in [Tak91]). Let (M,g) be a symmetric space and $x \in M$. Let G := I(M) be the isometry group of Mand $K \subset G$ be the subgroup of isometries which fix x.

Then there exists a unique analytic structure on G, which is compatible with the compactopen topology and makes G into a Lie group. With respect to this structure, K is a closed Lie subgroup of G, and

$$G/K \simeq M$$
$$gK \mapsto g(x) \tag{3}$$

is a G-equivariant diffeomorphism.

Example 7. S^n has group of isometries O(n+1), with identity component SO(n+1) and isotropy group $\simeq SO(n)$, thus we have $S^n \simeq SO(n+1)/SO(n)$.

Theorem 6 implies that we can write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for some suitable $\mathfrak{p} \subset \mathfrak{g}$ that has the same dimension as $T_x M$. And in fact, this can be made explicit in the following way:

Definition 8. Let $X \in T_x M$ and $\gamma^X(t) = \exp_x(tX)$ the respective geodesic passing through x. For $t \in \mathbb{R}$ let $s_t^X \in G$ be the symmetry around the point $\gamma^X(t)$. The map $p_t^X := s_{t/2}^X s_0 \in G$ is called *transvection in x in direction X*. Let P denote the set of all transvections in x and \mathfrak{p} its tangent space in the identity. The elements of \mathfrak{p} are called *infinitesimal transvections*.

Example 9. Figure 2 shows an element $X \in \mathfrak{p}$ in x. A map that moves points on S^2 along the arrows will then be an element in P.

In the case of $S^n = SO(n+1)/SO(n)$ we can compute P and p explicitly. Note that the



Figure 2: Example of an element $X \in \mathfrak{p}$

rotations in the (x_k, x_{n+1}) plane for $k \in \{1, \ldots, n\}$, i.e. the maps

$$A_{1} := \begin{pmatrix} \cos \varphi & 0 & 0 & \dots & 0 & -\sin \varphi \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \sin \varphi & 0 & 0 & \dots & 0 & \cos \varphi \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \varphi \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -\sin \varphi \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & \sin \varphi & 0 & \dots & 0 & \cos \varphi \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \cos \varphi \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \cos \varphi \end{pmatrix} = \exp \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sin \varphi & \cos \varphi \end{pmatrix} = \exp \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sin \varphi & \cos \varphi \end{pmatrix} = \exp \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sin \varphi & \cos \varphi \end{pmatrix}$$

are elements in P. Thus, $B_k \in \mathfrak{p}$ for $k \in \{1, \ldots, n\}$. dim $\mathfrak{p} = n$, thus (B_1, \ldots, B_n) is a

basis of \mathfrak{p} . In short, we have

$$\mathfrak{g} = \mathfrak{so}(n+1) = \begin{pmatrix} \mathfrak{so}(n) & 0\\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & v\\ -v^T & 0 \end{pmatrix} = \mathfrak{k} \oplus \mathfrak{p}.$$
(4)

Thus, in the case of S^n we found that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In fact, that is the case for all symmetric spaces. This decomposition carries additional algebraic structure, which is the key to the classification of all symmetric spaces and will be described in the following theorem:

Theorem 10 (Theorem 3.3 in [Hel01], Theorem 8.1.4 in [Wol11], Section 2.2 in [Tak91]). Let (M, g) be a symmetric space, $x \in M$, \mathfrak{k} the Lie algebra of the isotropy group and \mathfrak{p} the infinitesimal transvections. Then

$$\operatorname{Lie}(I(M)) =: \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{5}$$

and

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}. \tag{6}$$

Let s be the symmetry of M in x. Then

$$s = s^* : \mathfrak{X}(M) \to \mathfrak{X}(M)$$

$$X \mapsto ds^{-1}X(s(\cdot))$$
(7)

acts as follows: $s|_{\mathfrak{k}} = 1, \ s|_{\mathfrak{p}} = -1.$

Definition 11. A triple (\mathfrak{g}, s, B) of

- 1. \mathfrak{g} a Lie algebra,
- 2. $s : \mathfrak{g} \to \mathfrak{g}$ a degree two automorphism, such that the $\mathfrak{k} := \{X \in \mathfrak{g} : s(X) = X\}$ contains no non-zero ideal, and
- 3. B an s-invariant, $ad(\mathfrak{k})$ -invariant positive definite bilinear form on \mathfrak{g}

is called (abstract) orthogonal involutive Lie algebra. s is called Cartan involution.

The importance of orthogonal involutive Lie algebras comes from the fact that they arise as algebraic structures associated to symmetric spaces, as stated in the following theorem:

Theorem 12. Let (M, g) a symmetric space, and \mathfrak{g} and s as in Theorem 10.

Denote by B the inner product on $T_x M \simeq \mathfrak{p}$ and let B' be any $\mathrm{ad}(\mathfrak{k})$ -invariant positive definite bilinear form on \mathfrak{k} . Define $Q = B' \oplus B$.

Then (\mathfrak{g}, s, Q) is an orthogonal involutive Lie algebra.

Thus, it is clear that a classification of orthogonal involutive Lie algebras will be helpful for a classification of symmetric spaces. This will be made precise in the rest of the section. First note how we can go back from orthogonal involutive Lie algebras to symmetric spaces:

Theorem 13 (Theorems 8.3.6 and 8.3.7 in [Wol11]).

- 1. Let (\mathfrak{g}, s, Q) be an orthogonal involutive Lie algebra. Then there exists a symmetric space (M, g) such that (\mathfrak{g}, s, Q) is a subalgebra¹ of the orthogonal involutive Lie algebra associated by means of Theorem 12.
- 2. Let $(\mathfrak{g}, s, Q) \subset (\mathfrak{g}', s', Q')$ be orthogonal involutive Lie algebras. Let M and M' be the corresponding simply connected symmetric spaces. Then M is isometric to M'.

Proof.

1. Let (\mathfrak{g}, s, Q) be an abstract orthogonal involutive Lie algebra. Let \tilde{G} be the (up to isomorphism) uniquely determined simply connected Lie group with Lie algebra \mathfrak{g} . Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of \mathfrak{g} into (± 1) -eigenspaces of s. Let $\tilde{K} \subset \tilde{G}$ be the unique connected Lie subgroup of \tilde{G} with Lie algebra \mathfrak{k} .

Define $M := \tilde{G}/\tilde{K}$.

To construct the metric and symmetry on M, define $\tilde{Z} := \{g \in \tilde{G} \text{ such that } g : M \to M \text{ is the identity}\}$. \tilde{Z} is discrete, because \mathfrak{k} contains by assumption no nonzero ideal of \mathfrak{g} . Then, for $G_0 = \tilde{G}/\tilde{Z}$ and $K_0 = \tilde{K}/\tilde{Z}$, we have $M = G_0/K_0$, and G_0 acts effectively on M. Also, we still have $\operatorname{Lie}(G_0) = \mathfrak{g}$, $\operatorname{Lie}(K_0) = \mathfrak{k}$.

Let $o = eK_0 \in G_0/K_0$, then $\mathfrak{p} \simeq T_o M$ and $B := Q|\mathfrak{p}$ defines a positive definite inner product on $T_o M$. $\mathrm{ad}(\mathfrak{k})$ -invariance of Q translates this to a Riemannian metric on all of M.

s induces an automorphism on G_0 and therefore also on G_0/K_0 . s is an isometry of M with s(o) = o and $ds_o = -$ Id, thus it is a global symmetry as per definition 1.

2. See proof of Theorem 8.3.7 in [Wol11].

We can now explain the decomposition theory for symmetric spaces:

Definition 14.

- 1. An OILA (\mathfrak{g}, s, Q) is called *euclidean*, if $[\mathfrak{p}, \mathfrak{p}] = 0$. It is called *irreducible* if it is not euclidean and if the Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ acts irreducibly on \mathfrak{p} .
- 2. A Riemannian manifold is called *reducible* if its Riemannian universal cover M is isometric to the Riemannian product of at least two Riemannian manifolds of dimension ≥ 1 . Otherwise, M is called irreducible.

Theorem 15 (Theorem 8.2.4 of [Wol11]). Let (\mathfrak{g}, s, Q) be an orthogonal involutive Lie algebra. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_t$ direct sum of ideals, unique up to permutation, where

1. the \mathfrak{g}_i are s-invariant,

 $^{{}^{1}(\}mathfrak{g}, s, Q) \subset (\mathfrak{g}', s', Q')$ is an orthogonal involutive Lie *sub*algebra, if $\mathfrak{g} \subset \mathfrak{g}'$ is a subalgebra and s' and Q' are the restrictions of s and Q, such that $\mathfrak{p}' = \mathfrak{p}$ holds.

- 2. $(\mathfrak{g}_0, s|_{\mathfrak{g}_0}, Q|_{\mathfrak{g}_0})$ is euclidean,
- 3. for i > 0, \mathfrak{g}_i is semisimple and $(\mathfrak{g}_i, s|_{\mathfrak{g}_i}, Q|_{\mathfrak{g}_i})$ is irreducible.

Theorem 16 (Theorem 8.3.8 of [Woll1]). Let M be a simply connected Riemannian symmetric space. Then we have a decomposition $M = M_0 \times M_1 \times \cdots \times M_t$, unique up to permutation, such that M_0 is a euclidean space and M_i is simply-connected and irreducible for i > 0.

The M_i correspond to the factors \mathfrak{g}_i in the decomposition of the OILA of M.

Thus, in order to classify simply-connected Riemannian symmetric spaces, we are left with the task of classifying irreducible OILAs. Note that the restriction for M to be simplyconnected is not a strong one, as Theorem 8.3.12 of [Wol11] shows that any connected Riemannian symmetric space M is given as \tilde{M}/Γ for some simply-connected Riemannian symmetric space \tilde{M} and a finite subgroup $\Gamma \subset I(\tilde{M})$.

Duality allows to reduce this classification to the case of compact OILAs, as demonstrated in the following:

Definition 17. Let (\mathfrak{g}, s, Q) be an OILA with s-eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{p}^* := \sqrt{-1}\mathfrak{p} \subset \mathfrak{p}^{\mathbb{C}}$ and $\mathfrak{g}^* := \mathfrak{k} \oplus \mathfrak{p}^*$. Let s^* and Q^* be the obvious extensions to \mathfrak{g}^* .²

We then call $(\mathfrak{g}, s, Q)^* := (\mathfrak{g}^*, s^*, Q^*)$ the dual of (\mathfrak{g}, s, Q) .

Theorem 18 (Lemma 8.2.6 in [Wol11]). Let (\mathfrak{g}, s, Q) an OILA and $(\mathfrak{g}, s, Q)^*$ its dual.

- 1. $(g, s, Q)^{**} = (g, s, Q).$
- 2. If $(\mathfrak{g}, s, Q) = \sum (\mathfrak{g}_i, s_i, Q_i)$ is the decomposition of Theorem 15, then $(\mathfrak{g}, s, Q)^* = \sum (\mathfrak{g}_i, s_i, Q_i)^*$ is the decomposition of the dual OILA.
- 3. $(\mathfrak{g}, s, Q) \simeq (\mathfrak{g}, s, Q)^*$ if and only if the OILA is euclidean.
- 4. (\mathfrak{g}, s, Q) is irreducible if and only if $(\mathfrak{g}, s, Q)^*$ is irreducible.
- 5. If (\mathfrak{g}, s, Q) is irreducible, then precisely one of \mathfrak{g} and \mathfrak{g}^* is compact.

Example 19. Let us compute the dual space of $S^n = SO(n+1)/SO(n)$. We have the following Lie group isomorphism:

$$\mathfrak{so}(p+q)^* = \mathfrak{k} \oplus \mathfrak{p}^* \to \mathfrak{so}(p,q) \tag{8}$$
$$\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & iX_2 \\ -iX_2^T & 0 \end{pmatrix} \mapsto \begin{pmatrix} X_1 & X_2 \\ X_2^T & 0 \end{pmatrix}.$$

 $s^* : \mathfrak{so}(n+1)^* \to \mathfrak{so}(n+1)^*$ still has $s^*|_{\mathfrak{k}} = 1$, $s^*|_{\mathfrak{p}^*} = -1$, thus the dual space of $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ is by the construction from Theorem 13:

$$(S^n)^* := \mathrm{SO}_0(n,1) / \mathrm{SO}(n).$$
 (9)

²s extends to a unique complex-linear automorphism $s^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, which in turn restricts to \mathfrak{g}^* . $Q_{\mathfrak{p}^*}$ is defined by $Q_{\mathfrak{p}^*}(\sqrt{-1}X, \sqrt{-1}Y) := Q(X, Y)$.



Figure 3: Two models of the hyperbolic space $\mathbb{R}H^2$: the upper sheet of the hyperboloid (left) and the Poincaré disk (right)

This space is called *real hyperbolic space* and is, as expected, noncompact.

For n = 2, the space SO₀(2, 1)/SO(2) is called the *real hyperbolic plane*. It has no isometric embedding into \mathbb{R}^3 and no isometric equivariant embedding into \mathbb{R}^m for any $m \in \mathbb{N}$. We will look now at one of several realizations of the hyperbolic plane.

Let

$$Q^{+} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -x_1^2 - x_2^2 + x_3^2 = 1, x_2 > 0\} \subset \mathbb{R}^3$$
(10)

the upper sheet of the two-sheeted hyperboloid in \mathbb{R}^3 , as shown in figure 3.

Then $SO_0(2,1)$ acts transitively on Q^+ , and the stabilizer of the point $(0,0,1) \in Q^+$ is precisely

$$\operatorname{Stab}_{(0,0,1)} = \begin{pmatrix} \operatorname{SO}(2) & 0\\ 0 & 1 \end{pmatrix},\tag{11}$$

thus we have a bijection $Q^+ \stackrel{f}{\simeq} SO_0(2,1)/SO(2)$. The pulled back metric g of $SO_0(2,1)/SO(2)$ to Q^+ takes the form

$$f^*g(v,w) = \operatorname{arcosh}(v_1w_1 + v_2w_2 - v_3w_3) \text{ for } v, w \in Q^+.$$
(12)

Another realisation of the hyperbolic plane is the Poincaré disk $D^2 \subset \mathbb{R}^2$ (see figure 3) given by the map

$$Q^+ \mapsto D^2$$

 $(x_1, x_2, x_3) \to \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3}\right).$ (13)

One may now classify all compact OILAs and arrives at a classification of Riemannian symmetric spaces, as first done by Élie Cartan, see [Wol11, Theorem 8.11.4], [BCO16, Tables A.1, A.2, A.3, A.4].

2 Group Actions on Manifolds

In what follows, (M, g) is a Riemannian manifold and G a Lie group, acting smoothly on M by isometries.

2.1 Cohomogeneity One Actions

For $p \in M$ we denote by $G \cdot p$ the orbit of p under G, and by M/G the set of orbits of the G-action.

Definition 20.

- 1. Two orbits $G \cdot p$ and $G \cdot q$ are *equivalent*, if the stabilisers G_p and G_q are conjugate in G. In this case, we write $G \cdot p \sim G \cdot q$, and denote the equivalence class of $G \cdot p$ by $[G \cdot p]$. Denote by $\mathcal{O} = (M/G)/_{\sim}$ the set of orbit types.
- 2. We define the partial ordering \leq on \mathcal{O} by: $[G \cdot p] \leq [G \cdot q]$ if and only if G_q is conjugate to a subgroup of G_p in G.
- 3. Let $[G \cdot p] \in \mathcal{O}$ be a maximal element with respect to \leq . Then $G \cdot p$ is called *principal* orbit. An orbit that is not diffeomorphic to a principal orbit is called *exceptional* orbit. An orbit with dimension smaller than that of a principal orbit is called *singular* orbit.
- 4. The codimension of a principal orbit is called *cohomogeneity of the G-action*.

Remark 21. Note that all principal orbits have the same dimension. Thus the cohomogeneity of a G-action is a well-defined notion.

Example 22. As an example, consider the following SO(2)-action on S^2 :

$$SO(2) \times S^2 \to S^2 \tag{14}$$
$$(A, p) \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \cdot p.$$

Figure 4 shows the orbits of the points (1,0,0) and $(\sqrt{3}/2,0,1/2)$ in blue, which are principal orbits. It also shows the orbit of the north pole (0,0,1) in green, which is a singular orbit consisting of a single point.

In this example, the points whose orbit types are principal form an open and dense subset of S^2 —the only singular orbits are SO(2) \cdot (0,0,1) and SO(2) \cdot (0,0,-1). In fact, this is the case for *all* isometric group actions, see Proposition 2.2.4 of [BCO16].

2.2 Polar Actions

Definition 23 (Section 2.3.1 in [BCO16]). Let G be a closed subgroup of Isom(M).

1. Let $\Sigma \subset M$ be a connected, complete, embedded submanifold. Σ is called a *section* if



- Figure 4: Principal orbits and singular orbits for the SO(2)-action on S^2 that rotates the (x, y)-plane.
 - a) $\Sigma \cap G \cdot p \neq \emptyset$ for all $p \in M$ and
 - b) $\Sigma \perp (G \cdot p)$ for all orbits $G \cdot p$.
 - 2. If there exists a section for the G-action on M, then the action is called *polar*.
 - 3. If there exists a flat section for the G-action on M, then the action is called hyperpolar.

The relation between polar actions and cohomogeneity one actions is explained by means of the following theorem:

Theorem 24 (Corollary 2.13 of [HPTT95]). Let M = G/K be a symmetric space, where G is a compact, semisimple Lie group. Let $H \subset G$ be a closed connected subgroup acting by left-multiplication on M with cohomogeneity one. Then the action is hyperpolar.

2.3 Structure Theory of Real Lie Algebras

2.3.1 Restricted Root Decomposition

Let M = G/K be a symmetric space of noncompact type, $\mathfrak{g} = \text{Lie}(G)$, $\mathfrak{k} = \text{Lie}(K)$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the decomposition induced by the involution

$$\theta: \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \to \mathfrak{g}$$

$$X + Y \mapsto X - Y$$
(15)

from Theorem 10. Let B be the Killing form on \mathfrak{g} , then

$$\begin{array}{l} \langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \\ (X, Y) \mapsto -B(X, \theta Y) \end{array}$$

$$(16)$$

defines a positive definite inner product on \mathfrak{g} and θ is the algebra's Cartan involution. *Definition* 25. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace of \mathfrak{p} with dual space \mathfrak{a}^* . rank $(M) := \dim(\mathfrak{a})$ is called the *rank of* M. For $\alpha \in \mathfrak{a}^*$ let

$$\mathfrak{g}_{\alpha} = \{ X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a} \}.$$
(17)

For $\alpha \neq 0$ and $\mathfrak{g}_{\alpha} \neq 0$, α is called *restricted root* and \mathfrak{g}_{α} is called *restricted root space* (of \mathfrak{g} with respect to \mathfrak{a}). Denote by Ψ the set of all restricted roots.

The direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha \right) \tag{18}$$

is called *restricted root space decomposition* (of \mathfrak{g} with respect to \mathfrak{a}).

A subset $\Lambda = \{\alpha_1, \ldots, \alpha_r\} \subset \Psi$ is called a set of *simple roots* of Ψ , if every $\alpha \in \Psi$ can be written as

$$\alpha = \sum_{i=1}^{r} c_i \alpha_i \tag{19}$$

with some integers $c_1, \ldots, c_r \in \mathbb{Z}$ either all nonpositive or all nonnegative. The subset

$$\Psi^{+} = \left\{ \alpha \in \Psi : \alpha = \sum_{i=1}^{r} c_i \alpha_i, c_1, \dots, c_r \ge 0 \right\}$$
(20)

is called the set of *positive restricted roots* of Ψ with respect to Λ .

Example 26 (Section VI.4 of [Kna02]). Let $p \ge q$, p + q = n, and consider the group

$$G = \mathrm{SO}_0(p,q) = \left\{ A \in \mathbb{R}^{n \times n} : A^T \begin{pmatrix} -I_p & 0\\ 0 & I_q \end{pmatrix} A = \begin{pmatrix} -I_p & 0\\ 0 & I_q \end{pmatrix}, \det A = 1 \right\}$$
(21)

with Lie algebra

$$\mathfrak{so}(p,q) = \left\{ X \in \mathbb{R}^{n \times n} : X^T \begin{pmatrix} -I_p & 0\\ 0 & I_q \end{pmatrix} + \begin{pmatrix} -I_p & 0\\ 0 & I_q \end{pmatrix} X = 0, \text{tr } X = 0 \right\}$$
$$= \left\{ \begin{pmatrix} X_1 & X_3\\ X_3^T & X_2 \end{pmatrix} \in \mathbb{R}^{n \times n} : \begin{array}{c} X_1 \in \mathfrak{so}(p), X_2 \in \mathfrak{so}(q), \\ X_3 \in \mathbb{R}^{p \times q} \text{ arbitrary} \end{array} \right\}$$
(22)

and the subgroup

$$K = \mathrm{SO}(p)\,\mathrm{SO}(q) = \left\{ A = \begin{pmatrix} A_1 & 0\\ 0 & A_2 \end{pmatrix} \in \mathbb{R}^{n \times n} : A_1 \in \mathrm{SO}(p), A_2 \in \mathrm{SO}(q) \right\},\tag{23}$$

with Lie algebra

$$\mathfrak{k} = \operatorname{Lie}(\operatorname{SO}(p)\operatorname{SO}(q)) = \left\{ X = \begin{pmatrix} X_1 & 0\\ 0 & X_2 \end{pmatrix} \in \mathbb{R}^{n \times n} : X_1 \in \mathfrak{so}(p), X_2 \in \mathfrak{so}(q) \right\},$$
(24)

and complement

$$\mathfrak{p} = \left\{ X = \begin{pmatrix} 0 & X_3 \\ X_3^T & 0 \end{pmatrix} \in \mathbb{R}^{n \times n} : X_3 \in \mathbb{R}^{p \times q} \text{ arbitrary} \right\}.$$
(25)

We choose the maximal abelian subspace

$$\mathfrak{a} = \left\{ X = \begin{pmatrix} 0 & X_3 \\ X_3^T & 0 \end{pmatrix} \in \mathbb{R}^{n \times n} : X_3 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_q \\ 0 & 0 & \dots & a_{q-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_2 & \dots & 0 & 0 \\ a_1 & 0 & \dots & 0 & 0 \end{pmatrix}, a_1, \dots, a_q \in \mathbb{R} \right\}.$$
(26)

Define $f_i \in \mathfrak{a}^*$ to be the linear functional which has value a_i on the element of \mathfrak{a} with non-zero entries a_1, \ldots, a_q denoted above. The restricted roots are then

$$\Psi = \{ \pm f_i \pm f_j : 1 \le i < j \le q \} \cup \{ f_i : 1 \le i \le q \}.$$
(27)

To ease notation, we denote

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I_{2,0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
 (28)

The restricted root spaces for $\pm f_i \pm f_j$ are one-dimensional and are given as sets of sparse matrices, with nonzero entries only in the rows and columns with indices p - j + 1, p - i + 1, p + i, p + j, given as follows:

$$\mathfrak{g}_{f_i-f_j} = \operatorname{span}\left\{ \begin{pmatrix} J & -I_{2,0} \\ -I_{2,0} & -J \end{pmatrix} \right\}, \qquad \mathfrak{g}_{-f_i+f_j} = \operatorname{span}\left\{ \begin{pmatrix} J & I_{2,0} \\ I_{2,0} & -J \end{pmatrix} \right\}, \\ \mathfrak{g}_{f_i+f_j} = \operatorname{span}\left\{ \begin{pmatrix} J & -I_{1,1} \\ -I_{1,1} & -J \end{pmatrix} \right\}, \qquad \mathfrak{g}_{-f_i-f_j} = \operatorname{span}\left\{ \begin{pmatrix} J & I_{1,1} \\ I_{1,1} & J \end{pmatrix} \right\}.$$
(29)

The restricted root spaces for $\pm f_i$ have dimension (p-q) and are sparse matrices with nonzero entries only in the rows and columns with indices 1 to p-q, p-i+1 and p+i:

$$\mathfrak{g}_{f_i} = \left\{ A_v^+ := \begin{pmatrix} 0 & v & -v \\ -v^T & 0 & 0 \\ -v^T & 0 & 0 \end{pmatrix} \right\}, \qquad \mathfrak{g}_{-f_i} = \left\{ A_v^- := \begin{pmatrix} 0 & v & v \\ -v^T & 0 & 0 \\ v^T & 0 & 0 \end{pmatrix} \right\}, \quad (30)$$

where $v \in \mathbb{R}^{p-q}$ is a column vector.

2.3.2 Langlands Decomposition of Parabolic Subalgebras

Definition 27.

1. A maximal solvable subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is called *Borel subalgebra*.

2. A subalgebra $\mathfrak{q} \subset \mathfrak{g}$ is called *parabolic*, if it contains a Borel subalgebra.

Example 28 (Parabolic subalgebras defined by a set of restricted roots). Consider a Lie algebra \mathfrak{g} with its restricted root space decomposition, given in 18. Denote by

$$\mathfrak{n} = \bigoplus_{\alpha \in \Psi^+} \mathfrak{g}_{\alpha}.$$
(31)

It follows from $[\mathfrak{g}_{\alpha},\mathfrak{g}_{\beta}] \subset \mathfrak{g}_{\alpha+\beta}$ that \mathfrak{n} is a nilpotent subalgebra of \mathfrak{g} . The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$ is called *Iwasawa decomposition*. It induces a diffeomorphism $K \times A \times N \to G$.

 $\mathfrak{g}_0 \oplus \mathfrak{n}$ is a Borel subalgebra of \mathfrak{g} . Parabolic subalgebras turn out to be of the form

$$\mathfrak{q} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha\right) \tag{32}$$

for $\Psi^+ \subset \Gamma \subset \Psi$.

We now investigate how one may choose Γ so that the subvectorspace \mathfrak{q} is a subalgebra of \mathfrak{g} . To this end, let $\Phi \subset \Lambda$ and Ψ_{Φ} the root system generated by Φ and $\Psi_{\Phi}^{+} = \Psi_{\Phi} \cap \Psi^{+}$. Define

$$\mathfrak{l}_{\Phi} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Psi_{\Phi}} \mathfrak{g}_{\alpha}\right) \text{ and } \mathfrak{n}_{\Phi} = \bigoplus_{\alpha \in \Psi^+ \setminus \Psi_{\Phi}^+} \mathfrak{g}_{\alpha}.$$
(33)

Then

$$\mathfrak{q}_{\Phi} = \mathfrak{l}_{\Phi} \oplus \mathfrak{n}_{\Phi} \tag{34}$$

is a parabolic subalgebra, called the *parabolic subalgebra of* \mathfrak{g} associated with the subset Φ of Λ . The decomposition from equation 34 is called the *Chevalley decomposition* of \mathfrak{q}_{Φ} .

Let

$$\mathfrak{a}_{\Phi} = \bigcap_{\alpha \in \Phi} \operatorname{Ker} \alpha \subset \mathfrak{a}, \tag{35}$$

$$\mathfrak{m}_{\Phi} = \mathfrak{l}_{\Phi} \ominus \mathfrak{a}_{\Phi},\tag{36}$$

where \ominus is taken with respect to the bilinear form $\langle \cdot, \cdot \rangle$ defined in line 16, i.e. $\mathfrak{l}_{\Phi} = \mathfrak{a}_{\Phi} \oplus \mathfrak{m}_{\Phi}$ and $\mathfrak{a}_{\Phi} \perp \mathfrak{m}_{\Phi}$. The direct sum decomposition

$$\mathfrak{q}_{\Phi} = \mathfrak{m}_{\Phi} \oplus \mathfrak{a}_{\Phi} \oplus \mathfrak{n}_{\Phi} \tag{37}$$

is called *Langlands Decomposition* of q_{Φ} .

In fact, up to conjugation, all parabolic subalgebras are of the form q_{Φ} for some $\Phi \subset \Lambda$. This is made precise in the following theorem:

Theorem 29 (Theorem 13.2.1 of [BCO16]). Let \mathfrak{g} be a semisimple Lie algebra with restricted simple roots Λ .

- 1. Every parabolic subalgebra \mathfrak{q} of \mathfrak{g} is conjugate in \mathfrak{g} to \mathfrak{q}_{Φ} for some subset Φ of Λ .
- 2. Two parabolic subalgebras \mathfrak{q}_{Φ_1} and \mathfrak{q}_{Φ_2} of \mathfrak{g} are conjugate in the full automorphism group $\operatorname{Aut}(\mathfrak{g})$ of \mathfrak{g} if and only if there exists and automorphism F of the Dynkin diagram associated with Λ such that $F(\Phi_1) = \Phi_2$.

2.4 Classification of Cohomogeneity One Actions on Symmetric Spaces of Noncompact Type without Exceptional Orbit

Let M = G/K be a symmetric space with G semisimple, and let H act on M through isometries. Then $H \subset G$ up to covering, and H acts on M by left-multiplication. In what follows we want to find all H such that this action has cohomogeneity one. If M is of noncompact type, one has the following result:

Theorem 30 (Section 1 of [BB01]). Let M = G/K be a symmetric space of noncompact type with G semisimple and let H act on M through isometries with cohomogeneity one. Then one of the two cases applies:

- 1. The action has no singular orbit.
- 2. The action has exactly one singular orbit.

Remark 31. Theorem 30 is not true for symmetric spaces of compact type, as example 22 shows. In fact, isometric group actions of cohomogeneity one on symmetric spaces of compact type always have *at least* two exceptional (i.e. not diffeomorphic to a principal orbit) orbits, both of which may be singular. (Proposition 5 of [Ver03])

An isometric group action on a homogenous space without singular orbit can equivalently be considered as a homogenous Riemannian foliation.

Definition 32. Two foliations \mathfrak{F} and \mathfrak{G} of a Riemannian manifold M are called *isometrically* congruent if there exists an isometry $f: M \to M$ that maps leaves of \mathfrak{F} onto leaves of \mathfrak{G} .

Up to isometrical congruence, homogenous Riemannian foliations on irreducible M fall into two classes: foliations of type $\mathfrak{F}_{\mathfrak{l}}$ for $\mathfrak{l} \in \mathbb{RP}^{r-1}$, where r is the rank of M, and foliations of type \mathfrak{F}_i for $i \in \{1, \ldots, r\}$.

In the following let $\{\alpha_1, \ldots, \alpha_r\} \subset \Psi$ be a set of simple restricted roots for G/K.

Foliations of Type \mathfrak{F}_l

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and G = KAN be the Iwasawa decompositions of \mathfrak{g} and G respectively. Let $\mathfrak{l} \subset \mathfrak{a}$ be a one-dimensional subspace, and define

$$\mathfrak{s}_{\mathfrak{l}} = \mathfrak{n} \oplus (\mathfrak{a} \ominus \mathfrak{l}) \subset \mathfrak{a} \oplus \mathfrak{n}.$$

$$(38)$$

Let $\exp(\mathfrak{s}_{\mathfrak{l}}) = S_{\mathfrak{l}} \subset AN$ the corresponding Lie subgroup of codimension one.

 $S_{\mathfrak{l}}$ then acts with cohomogeneity one on AN, and because of $M = G/K = (KAN)/K \simeq AN$, this defines a cohomogeneity one action on M.

Example 33. Consider the hyperbolic 2-space $(S^2)^* = \mathbb{RH}^2 = SO_0(2,1)/SO(2)$ from example 19. As seen in example 26, the restricted root decomposition is in this case

$$\mathfrak{so}_{0}(2,1) = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & w \\ 0 & w & 0 \end{pmatrix}}_{\mathfrak{a}=\mathfrak{g}_{0}} \oplus \underbrace{\begin{pmatrix} 0 & -a & a \\ a & 0 & 0 \\ a & 0 & 0 \end{pmatrix}}_{\mathfrak{g}_{f}} \oplus \underbrace{\begin{pmatrix} 0 & a & a \\ -a & 0 & 0 \\ a & 0 & 0 \end{pmatrix}}_{\mathfrak{g}_{-f}}, \tag{39}$$

where $f \in \mathfrak{a}^*$ is the linear functional that sends the above element of \mathfrak{a} to the value w. This gives rise to the Iwasawa decomposition

Since \mathfrak{a} is one-dimensional, the only choice for $\mathfrak{l} \subset \mathfrak{a}$ is $\mathfrak{l} = \mathfrak{a}$. We receive the cohomogeneity one action of $\exp(\mathfrak{n}) = N$ on AN. Figure 33 shows some orbits under this action.



Figure 5: Orbits of the points $(0, \kappa, \sqrt{\kappa^2 + 1})$ for different values of κ on \mathbb{RH}^2 under the action of N on AN in the hyperboloid model (left) and the Poincaré disk model (right).

Example 34. Consider the hyperbolic 3-space with its Poincaré ball model in analogy to the above described Poincarè disk model. Again, we have a unique choice $\mathfrak{l} \subset \mathfrak{a}$ and the respective foliation is displayed in figure 34.



Figure 6: The action of N on AN for the case of $\mathbb{R}H^3$ in the Poincaré ball model.

Two foliations $\mathfrak{F}_{\mathfrak{l}}$ and $\mathfrak{F}_{\mathfrak{l}'}$ of this type may be isometrically congruent. It was shown in [BT03a, Theorem 3.5], that $\mathfrak{F}_{\mathfrak{l}}$ and $\mathfrak{F}_{\mathfrak{l}'}$ are isometrically congruent to each other if and

only if there exists a symmetry $P \in \mathfrak{A}$ of the Dynkin diagram of M with $P(\mathfrak{l}) = P(\mathfrak{l}')$. Here, the action of $P \in \mathfrak{A}$ on \mathbb{RP}^{r-1} is given as the projectivisation of the action P: $\operatorname{span}(\alpha_1, \ldots, \alpha_r) \to \operatorname{span}(\alpha_1, \ldots, \alpha_r)$.

Foliations of Type \mathfrak{F}_i

For $\xi \in \mathfrak{g}_{\alpha_i}$ a unit vector, define

$$\mathfrak{s}_{\xi} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathbb{R}\xi) \subset \mathfrak{a} \oplus \mathfrak{n} \tag{41}$$

and $\exp(\mathfrak{s}_{\xi}) = S_{\xi} \subset AN$. As before, S_{ξ} acts with cohomogeneity one on $M \simeq AN$. Different choices of ξ lead to isometrically congruent foliations, to be precise: If $\eta \in \mathfrak{g}_{\alpha_i}$ is another unit vector in the same root space, then there exists an isometry $f \in Z_{\mathfrak{k}}(\mathfrak{a})$ mapping the orbits of \mathfrak{F}_{ξ} onto the orbits of \mathfrak{F}_{η} , cf. [BT03b, Lemma 4.1]. Up to isometric congruency the foliations thus depend only on \mathfrak{g}_{α_i} , and not on the choice of unit vector therein. We thus denote by \mathfrak{F}_i the foliation generated by any unit vector in \mathfrak{g}_{α_i} .

Example 35. In analogy to Example 33, Figure 35 shows some orbits of the action of $\begin{pmatrix} 0 & a & -a \end{pmatrix}$

$$A = \exp \begin{pmatrix} 0 & a & -a \\ a & 0 & 0 \\ a & 0 & 0 \end{pmatrix}$$
 on $AN = \mathbb{RH}^2$.





Figure 7: Orbits of the points $\left(-\kappa, 0, \sqrt{\kappa^2 + 1}\right)$ for different values of κ on \mathbb{RH}^2 under the action of A on AN.

Example 36. In the case of $\mathbb{R}H^3$ we have different choices for unit vectors $\xi \in \mathfrak{a}$ leading to isometrically congruent foliations. Note that $\mathfrak{s}_{\xi} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathbb{R}\xi)$ in this case is precisely the Borel algebra of $\mathbb{R}H^2$, i.e. we can see an embedded $\mathbb{R}H^2 \subset \mathbb{R}H^3$ as the orbit of $o = e \operatorname{SO}(3)$ under the action of S_{ξ} , compare figure 8.



Figure 8: The foliation of type \mathfrak{F}_1 of $\mathbb{R}H^3$.

However, there are even more isometric congruencies between foliations of this kind. It is shown in [BT03a, Theorem 4.8] (cf. [BCO16, Proposition 13.4.4]) that \mathfrak{F}_i and \mathfrak{F}_j are

isometrically congruent to each other if and only if there exists a symmetry $P \in \mathfrak{A}$ of the Dynkin diagram of G/K with $P(\alpha_i) = \alpha_j$.

The Moduli Space of All Smooth Foliations

Taking the foliations of both types together we get the following result:

Theorem 37 (Theorem 13.4.5 in [BCO16]). Let M be a connected irreducible Riemannian symmetric space of noncompact type and with rank r. The moduli space \mathfrak{M}_F of all noncongruent homogenous codimension one foliations of M is isomorphic to the orbit space of the action of the symmetry group of the Dynkin diagram of M, \mathfrak{A} , on $\mathbb{RP}^{r-1} \cup \{1, \ldots, r\}$:

$$\mathfrak{M}_F \simeq (\mathbb{R}\mathbb{P}^{r-1} \cup \{1, \dots, r\})/\mathfrak{A}.$$
(42)

2.5 Actions With Exceptional Orbit

2.5.1 Hyperbolic Spaces

The noncompact symmetric spaces of rank 1 are called *hyperbolic spaces*. Following from the classification of compact symmetric spaces and the duality principle (cf. Theorem 18) the complete list of hyperbolic spaces is: $\mathbb{R}H^n = \mathrm{SO}_0(n, 1)/\mathrm{SO}(n)$, $\mathbb{C}H^n = \mathrm{SU}(n, 1)/\mathrm{S}(\mathrm{U}(n) \mathrm{U}(1))$, $\mathbb{H}H^n = \mathrm{Sp}(n, 1)/\mathrm{Sp}(1) \mathrm{Sp}(n)$, $\mathbb{O}H^2 = F_4^{-20}/\mathrm{Spin}(9)$.

Real Hyperbolic Space

Theorem 38 (Theorem 6.1 of [BT13]). Every cohomogeneity one action with singular orbit on the real hyperbolic space $\mathbb{R}H^n = SO_0(n, 1)/SO(n)$ is orbit equivalent to the following action for some $k \in \{0, ..., n-2\}$:

The action of $SO_0(k, 1) \times SO(n-k) \subset SO_0(n, 1)$ whose singular orbit is a totally geodesic $\mathbb{R}H^k \subset \mathbb{R}H^n$.

Example 39. From Example 26 follows that the Iwasawa decomposition for $\mathbb{R}H^3$ is

and (up to orbit equivalence) Theorem 38 gives two different cohomogeneity one actions on $\mathbb{R}H^3$ which are given by left multiplication of $K_0A = \exp(\mathfrak{k}_0 \oplus \mathfrak{a})$ and K = SO(3)respectively.

In the first case, the singular orbit of the action is an $H \cdot o = \mathbb{R}H^1$ (cf. Figure 9). In the second case, the singular orbit is only a point.



Figure 9: Orbits of the action of K_0A on $\mathbb{R}H^3$. The singular orbit (right) is a copy of $\mathbb{R}H^1$, the principal orbit has dimension 2.

2.5.2 Symmetric Spaces of Higher Rank

Let G/K be an irreducible symmetric space of noncompact type with rank $r \ge 2$. Let H act by isometries with cohomogeneity one on G/K with a singular orbit.

Theorem 40 ([Mos61], p.374 in [BCO16]). Let L be a connected proper maximal subgroup of G with $H \subset L$ and denote by l its Lie algebra. Then one of the following two is true:

- 1. It is reductive, the actions of H and L are orbit equivalent and $W = L \cdot o$ is a totally geodesic submanifold.
- 2. l is parabolic and the actions of H and L are not orbit equivalent.

The cohomogeneity one actions in the first case have been classified for all noncompact spaces in [BT04a]. It has been shown in [BT13] that in the second case all groups H arise from *canonical extension* or *nilpotent construction*. For the rest of the chapter the two construction methods will be introduced.

Canonical Extension

Let $\mathfrak{q}_{\Phi} \subset \mathfrak{g}$ parabolic with Langlands decomposition (cf. line 37)

$$\mathfrak{q}_{\Phi} = \mathfrak{m}_{\Phi} \oplus \mathfrak{a}_{\Phi} \oplus \mathfrak{n}_{\Phi}. \tag{45}$$

Definition 41. Let $\mathfrak{g}_{\Phi} = [\mathfrak{m}_{\Phi}, \mathfrak{m}_{\Phi}]$ and G_{Φ} the closed connected subgroup of G with Lie algebra \mathfrak{g}_{Φ} . The semisimple Riemannian symmetric space $B_{\Phi} = M_{\Phi} \cdot o = G_{\Phi} \cdot o$ is called *boundary component of* G/K with respect to Φ . We denote by F_{Φ} the corresponding submanifold $F_{\Phi} := L_{\Phi} \cdot o = B_{\Phi} \times \mathbb{E}^{r-\Phi}$, where $\mathbb{E}^{r-|\Phi|}$ denotes a euclidean space of dimension $r - |\Phi|$.

Proposition 42. Let G/K be a symmetric space of noncompact type, $\Phi \subset \Lambda$ a set of simple roots. Then the boundary component B_{Φ} is a symmetric space of noncompact type with restricted Dynkin diagram given by restricting the diagram of G/K to the roots Φ .

Example 43. Consider the rank 3 space $SO_0(3,4)/(SO(3)SO(4))$. It has simple roots $\Lambda = \{\alpha_1, \alpha_2, \alpha_3\}$ and restricted Dynkin diagram of type B_3 :



There are three boundary components of rank 2, corresponding to the choices $\Phi_3 = \{\alpha_1, \alpha_2\}, \ \Phi_1 = \{\alpha_2, \alpha_3\}, \ \text{and} \ \Phi_2 = \{\alpha_1, \alpha_3\}.$ According to the previous proposition the boundary components are the following symmetric spaces:

Φ	Dynkin diagram	Symmetric space
Φ_3	$0 \longrightarrow 0 \rightarrow 0 \longrightarrow 0$	$\operatorname{SL}(3,\mathbb{R})/\operatorname{SO}(3)$
Φ_1	$\circ \longrightarrow \circ \rightarrow \circ \longrightarrow \circ$	${\rm SO}_0(2,3)/({\rm SO}(2){\rm SO}(3))$
Φ_2	$\bigcirc \longrightarrow \bigcirc \bigcirc \rightarrow \circ \circ$	$(\operatorname{SL}(2,\mathbb{R})/\operatorname{SO}(2))^2$

Proposition 44 (Proposition 4.1 in [BT13]). Let M be a symmetric space of noncompact type, $\Phi \subset \Lambda$ a set of simple roots, \mathfrak{q}_{Φ} the corresponding parabolic subalgebra with Langlands decomposition $\mathfrak{q}_{\Phi} = \mathfrak{m}_{\Phi} \oplus \mathfrak{a}_{\Phi} \oplus \mathfrak{n}_{\Phi}$ and boundary component B_{Φ} .

Let \mathfrak{h}_{Φ} be the Lie algebra of a connected closed Lie subgroup $H_{\Phi} \subset G_{\Phi}$ acting on B_{Φ} with cohomogeneity one.

Let H be the connected closed subgroup of Q_{Φ} with Lie algebra

$$\mathfrak{h} := \mathfrak{h}_{\Phi} \oplus \mathfrak{a}_{\Phi} \oplus \mathfrak{n}_{\Phi}. \tag{46}$$

Then H acts on M with cohomogeneity one.

Example 45. $SO(3, 4)_0/(SO(3)SO(4))$ has three different boundary components, according to Example 43.

- 1. $B_{\Phi_3} = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$: The cohomogeneity one actions on this space were classified in [BT13, Theorem 6.2]. By Proposition 44 each of these has a canonical extension to a cohomogeneity one action on $\mathrm{SO}(3,4)_0/(\mathrm{SO}(3)\,\mathrm{SO}(4))$.
- 2. $B_{\Phi_1} = SO_0(2,3)/(SO(2)SO(3))$: The cohomogeneity one actions on this space were classified in [BT13, Theorem 6.3]. By Proposition 44 each of these has a canonical extension to a cohomogeneity one action on $SO(3,4)_0/(SO(3)SO(4))$.
- 3. $B_{\Phi_2} = \operatorname{SL}(2,\mathbb{R})/\operatorname{SO}(2) \times \operatorname{SL}(2,\mathbb{R})/\operatorname{SO}(2) = \mathbb{R}H^2 \times \mathbb{R}H^2$: The space is reducible, thus the cohomogeneity one actions on this space cannot be classified with the methods discussed here. Because of the low dimension of the isometry group $\operatorname{SL}(2,\mathbb{R}) \times \operatorname{SL}(2,\mathbb{R})$ one may explicitly compute all subgroups of the isometry group and verify in each case whether the subgroups act with cohomogeneity one.

As an example, consider $H_2 = A_{\{2,3\}} \times A_{\{1,2\}} N_{\{1,2\}}$ acting on $\mathbb{R}H^2 \times \mathbb{R}H^2$ with cohomogeneity one. Here, $A_{\{2,3\}}$ denotes the abelian subgroup in the Iwasawa decomposition for the boundary component $B_{\{2,3\}} \simeq \mathbb{R}H^2$, analogously for $B_{\{1,2\}} \simeq \mathbb{R}H^2$. The corresponding Lie algebra is

$$\mathfrak{h}_2 = \langle H^1 \rangle \oplus \langle H^3 \rangle \oplus \mathfrak{g}_{\alpha_3}, \tag{47}$$

where H^i are the dual vectors of α_i . The canonical extension of this action thus gives the Lie algebra

$$\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{n} \ominus \mathfrak{g}_{\alpha_1}, \tag{48}$$

i.e. the canonical extension yields the foliation \mathfrak{F}_1 defined in the previous section.

Nilpotent Construction

Let $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$ be a set of simple roots for \mathfrak{g} and $\Phi \subset \Lambda$. Let \mathfrak{q}_{Φ} be the parabolic subalgebra of \mathfrak{g} with Chevalley decomposition $\mathfrak{q}_{\Phi} = \mathfrak{l}_{\Phi} \oplus \mathfrak{n}_{\Phi}$ (cf. equation 34).

 \mathfrak{n}_{Φ} is endowed with a natural gradation which is described in the following. Denote by $H^1, \ldots, H^r \in \mathfrak{a}$ the dual basis of $\alpha_1, \ldots, \alpha_r$. Let

$$H^{\Phi} = \sum_{\alpha \in \Lambda \setminus \Phi} H^i \tag{49}$$

and $m_{\Phi} := \delta(H^{\Phi})$, where δ is the highest root in Ψ^+ . For $\nu \in \{1, \ldots, m_{\Phi}\}$ define

$$\mathfrak{n}_{\Phi}^{\nu} = \bigoplus_{\substack{\alpha \in \Psi^{+} \setminus \Psi_{\Phi}^{+} \\ \alpha(H^{\Phi}) = \nu}} \mathfrak{g}_{\alpha}.$$
(50)

Then

$$\mathfrak{n}_{\Phi} = \bigoplus_{\nu=1}^{m_{\Phi}} \mathfrak{n}_{\Phi}^{\nu} \tag{51}$$

is an $\operatorname{Ad}(K_{\Phi})$ -invariant gradation of \mathfrak{g} , where $K_{\Phi} = K \cap Q_{\Phi}$.

Proposition 46 (Proposition 13.6.4. in [BCO16], Proposition 4.3 in [BT13]). Assume that dim $\mathfrak{n}_{\Phi}^1 \geq 2$ and let \mathfrak{v} be a subspace of \mathfrak{n}_{Φ}^1 with dim $\mathfrak{v} \geq 2$. Define $\mathfrak{n}_{\Phi,\mathfrak{v}} = \mathfrak{n}_{\Phi} \ominus \mathfrak{v}$. Assume that

1. $N_{L_{\Phi}}^{0}(\mathfrak{n}_{\Phi,\mathfrak{v}}) := \{g \in L_{\Phi} : \mathrm{Ad}(g)\mathfrak{n}_{\Phi,\mathfrak{v}} \subset \mathfrak{n}_{\Phi,\mathfrak{v}}\}^{0} \text{ acts transitively on } F_{\Phi} = L_{\Phi} \cdot o,$

2.
$$N^0_{K_{\Phi}}(\mathfrak{n}_{\Phi,\mathfrak{v}}) = N^0_{K_{\Phi}}(\mathfrak{v})$$
 acts transitively on the unit sphere in \mathfrak{v} .

Then

$$H_{\Phi,\mathfrak{v}} := N_{L_{\Phi}}^{0}(\mathfrak{n}_{\Phi,\mathfrak{v}})N_{\Phi,\mathfrak{v}}$$

$$\tag{52}$$

acts on M with cohomogeneity one and $H_{\Phi,v} \cdot o$ is a singular orbit of this action containing F_{Φ} . Here $N_{\Phi,v}$ denotes the connected subgroup of G with Lie algebra $\mathfrak{n}_{\Phi,v}$.

We now give a sufficient condition to check if a subspace $\mathfrak{v} \subset \mathfrak{n}_{\Phi}^1$ satisfies the assumption of the previous proposition.

Lemma 47. Let H act on M = G/K by isometries where $G = \text{Isom}^0(M)$ and denote the Iwasawa decomposition of G by G = KAN. Assume that $\mathfrak{a} \oplus \mathfrak{n} \subset \mathfrak{h}$. Then H acts transitively on M. *Proof.* $AN \subset G$ is the unique connected subgroup of G with Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$, thus $AN \subset H$. AN acts transitively on M, therefor H does too.

Lemma 48. Let G be a Lie group acting on a compact Riemannian manifold M through isometries. Fix $x \in M$ and denote by

$$\Phi : \mathfrak{g} \to T_x M$$

$$X \mapsto \frac{d}{dt} \exp(tX) x|_{t=0}$$
(53)

the infinitesimal action of \mathfrak{g} on $T_x M$. Then G acts transitively if and only if Φ is surjective.

Proof. " \Rightarrow ": is clear.

" \Leftarrow ": Let $y \in M$ and $\Phi(V) \in T_x M$ such that $\exp_x(\Phi(V)) = y$. Then

$$y = \exp_x(\Phi(V)) = \exp(V) \cdot x$$

i.e. G acts transitively.

This allows us to translate the conditions from Proposition 46 into algebraic statements. For the first point from the proposition we find:

Lemma 49. Let the notation be as in Proposition 46. Denote by $\mathfrak{g}_{\Phi} = \mathfrak{k}_{B_{\Phi}} \oplus \mathfrak{a}_{B_{\Phi}} \oplus \mathfrak{n}_{B_{\Phi}}$ the Iwasawa decomposition for the boundary component $B_{\Phi} = G_{\Phi} \cdot o$. (Here $\mathfrak{g}_{\Phi} = [\mathfrak{l}_{\Phi}, \mathfrak{l}_{\Phi}]$ and G_{Φ} denotes the corresponding connected subgroup of G) If $\mathfrak{a}_{B_{\Phi}} \oplus \mathfrak{n}_{B_{\Phi}} \subset \mathfrak{n}_{\mathfrak{l}_{\Phi}}(\mathfrak{n}_{\Phi,\mathfrak{v}}) :=$ $\{X \in \mathfrak{l}_{\Phi} : [X, \mathfrak{n}_{\Phi,\mathfrak{v}}] \subset \mathfrak{n}_{\Phi,\mathfrak{v}}\}$, then $N^{0}_{L_{\Phi}}(\mathfrak{n}_{\Phi,\mathfrak{v}})$ acts transitively on $F_{\Phi} = L_{\Phi} \cdot o$.

For the second point from the proposition we find:

Lemma 50. Let the notation be as in Proposition 46. $N^0_{K_{\Phi}}(\mathfrak{n}_{\Phi,\mathfrak{v}})$ acts transitively on the unit sphere \mathbb{S} in \mathfrak{v} if and only if $[\mathfrak{n}_{\mathfrak{t}_{\Phi}}(\mathfrak{n}_{\Phi,\mathfrak{v}}), x] = x^{\perp}$ for some (and therefor any) $x \in \mathbb{S}$. (Here x^{\perp} denotes the orthogonal complement in \mathfrak{v} , i.e. $\mathfrak{v} \ominus x$)

Proof. " \Rightarrow ": is clear.

" \Leftarrow ": Let $x \in S$. $N^0_{K_{\Phi}}(\mathfrak{n}_{\Phi,\mathfrak{v}})$ acts on S by isometries through the adjoint representation. The induced infinitesimal action on T_xS is

$$\Phi: V \mapsto \frac{d}{dt} \exp(tV) x|_{t=0} = \frac{d}{dt} \operatorname{Ad}(\exp(tV)) x = [V, x] \in x^{\perp}$$
(54)

and the claim follows from Lemma 48.

Example 51. Consider as before $SO(3, 4)_0/(SO(3) SO(4))$. We listed all boundary components of rank 2 in Example 43. We now give one example of a cohomogeneity one actions arising from nilpotent construction with respect to the boundary component B_3 .

 $\Phi_3 = \{\alpha_1, \alpha_2\}$. In this case $m_3 = \delta(H^3) = 2$ and the gradation of \mathfrak{n}_3 is not trivial. The boundary component is $B_3 = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ with Iwasawa decomposition $\mathfrak{sl}(3, \mathbb{R}) = \mathfrak{so}(3) \oplus \mathfrak{a}_{B_3} \oplus \mathfrak{n}_{B_3}$ where $\mathfrak{n}_{B_3} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$. We have

$$\begin{split} \mathfrak{n}_{3}^{1} &= \mathfrak{g}_{\alpha_{1}+\alpha_{2}+\alpha_{3}} \oplus \mathfrak{g}_{\alpha_{2}+\alpha_{3}} \oplus \mathfrak{g}_{\alpha_{3}}, \\ \mathfrak{n}_{3}^{2} &= \mathfrak{g}_{\alpha_{1}+2\alpha_{2}+2\alpha_{3}} \oplus \mathfrak{g}_{\alpha_{2}+2\alpha_{3}} \oplus \mathfrak{g}_{\alpha_{1}+\alpha_{2}+2\alpha_{3}}, \\ \mathfrak{l}_{3} &= \mathfrak{g}_{\pm\alpha_{1}} \oplus \mathfrak{g}_{\pm\alpha_{2}} \oplus \mathfrak{g}_{\pm(\alpha_{1}+\alpha_{2})} \oplus \mathfrak{g}_{0}, \\ \mathfrak{k}_{3} &= \mathfrak{so}(3). \end{split}$$

Let $\mathfrak{v} := \mathfrak{n}_3^1$. One checks directly that $\mathfrak{n}_{B_3} \subset \mathfrak{n}_{\mathfrak{l}_3}(\mathfrak{n}_{3,\mathfrak{v}})$. $\mathfrak{a}_{F_3} \subset \mathfrak{n}_{\mathfrak{l}_3}(\mathfrak{n}_{3,\mathfrak{v}})$ holds because of $\mathfrak{a}_{B_3} \subset \mathfrak{g}_0$. It follows from Lemma 49 that $N^0_{L_3}(\mathfrak{n}_{3,\mathfrak{v}})$ acts transitively on F_3 .

To show that $N_{K_3}^0(\mathfrak{v})$ acts transitively on $\mathbb{S} \subset \mathfrak{v}$, fix some $x \in \mathfrak{g}_{\alpha_3}$ with |x| = 1. We have $\mathfrak{g}_{\pm \alpha_2}, \mathfrak{g}_{\pm (\alpha_1 + \alpha_2)} \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{v})$ and thus $\mathfrak{n}_{\mathfrak{k}_3}(\mathfrak{v}) \cap (\mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{-\alpha_2}) = (\mathfrak{k}_3 \cap \mathfrak{n}_{\mathfrak{g}}(\mathfrak{v})) \cap (\mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{-\alpha_2}) \neq \{0\}.$

To see this remember the root decomposition of SL(3):

$$\mathfrak{g}_{\alpha_2} = \mathfrak{g}_{e_2-e_3} = \operatorname{span} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\mathfrak{g}_{-\alpha_2} = \mathfrak{g}_{e_3-e_2} = \operatorname{span} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\mathfrak{so}(3) \cap (\mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{-\alpha_2}) = \operatorname{span} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

So let $0 \neq y \in \mathfrak{n}_{\mathfrak{k}_3}(\mathfrak{v}) \cap (\mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{-\alpha_2})$. Then $0 \neq [x, y] \in \mathfrak{g}_{\alpha_2 + \alpha_3}$.

Similarly we find $0 \neq z \in \mathfrak{n}_{\mathfrak{k}_3}(\mathfrak{v}) \cap (\mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{-(\alpha_1+\alpha_2)})$ such that $0 \neq [x, z] \in \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3}$.

Thus $[\mathfrak{n}_{\mathfrak{k}_3}(\mathfrak{n}_{3,\mathfrak{v}}), x] = x^{\perp}$ and therefore, by Lemma 50, $N^0_{K_3}(\mathfrak{v})$ acts transitively on S.

By means of Proposition 46 we see that $H_{3,\mathfrak{v}} := N_{L_3}^0(\mathfrak{n}_{3,\mathfrak{v}})N_{3,\mathfrak{v}}$ acts on M with cohomogeneity one. The singular orbit $H_{3,\mathfrak{v}} \cdot o$ has dimension

$$\dim(H_{3,\mathfrak{v}} \cdot o) = \dim F_3 + \dim N_{3,\mathfrak{v}} = 6 + 3 = 9.$$

The positive roots of the isometry Lie algebra of the singular orbit are

$$\mathfrak{g}_{\alpha_2+2\alpha_3},\mathfrak{g}_{\alpha_1},\mathfrak{g}_{\alpha_2},\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3},\mathfrak{g}_{\alpha_1+\alpha_2},\mathfrak{g}_{\alpha_1+2\alpha_2+2\alpha_3},\tag{55}$$

and writing $\tilde{\alpha}_3 := \alpha_2 + 2\alpha_3$, we see that this is the root system of the symmetric space $SL_4(\mathbb{R})/SO(4)$.

Note that this example appears in the classification of cohomogeneity one actions with totally geodesic singular orbit on $SO(3, 4)_0/(SO(3)SO(4))$, cf. [BCO16, Table 13.1], [BT04b, Theorem 3.3].

2.6 Outlook

Using the techniques layed out in the previous sections of the chapter, one may classify all cohomogeneity one actions on symmetric spaces of noncompact type of rank up to 2. This has been done for all such spaces of rank 1 and some spaces of rank 2. The classification results for spaces of rank 2 that appear in the literature can be found in [BT13] and [BDV15]. One example of a space on which the cohomogeneity one actions have not been classified is the following:

Open Problem 52. Classify cohomogeneity one actions on $SO(5, \mathbb{C})/SO(5)$.

For higher rank, the situation changes. It is still true that cohomogeneity one actions are classified in so far as that they are either smooth foliations, or are one of finitely many possibilities of actions by groups contained in a reductive subgroup, or arise from canonical extension or nilpotent construction.

However, in the case of rank ≥ 3 , boundary components may be reducible. In this case, the previously discussed classification results do not apply. We have encountered the simplest possible reducible boundary component in Example 45. The obvious problem is therefore:

Open Problem 53. Classify cohomogeneity one actions on $\mathbb{R}H^2 \times \mathbb{R}H^2$.

Once this has been achieved, one may again employ the standard constructions discussed in the previous section to classify cohomogeneity one actions on some rank 3 spaces. A good start would be:

Open Problem 54. Classify cohomogeneity one actions on $SO(3, 4)_0/(SO(3)SO(4))$.

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