Numerical approximations of harmonic 1-forms on real loci of Calabi-Yau manifolds

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Abstract: For applications, it is desirable to have Calabi-Yau manifolds of complex dimension three which contain a real three-dimensional submanifold with a nowhere vanishing harmonic 1-form. The harmonic equation depends on the Calabi-Yau metric, which is not known explicitly. Currently, one conjectural example of such a manifold based on the SYZ conjecture exists. In the talk, I will explain a second conjectural example motivated by numerical approximations using neural networks. The numerical approximation works by first computing an approximate Calabi-Yau metric, and then computing a harmonic form for this approximate metric. This is based on an approach by Donaldson for computing approximate Calabi-Yau metrics that I will briefly review. I will also compare the results with a proven non-example, i.e. one where a harmonic 1-form exists, but it is guaranteed to have zeros. Such examples are interesting for M-theory in Physics. This is based on arXiv:2405.19402, which is joint work with Mike Douglas, Yidi Qi, and Rodrigo Barbosa. Time permitting I will comment on using an approximate harmonic 1-form in a numerically verified proof to rigorously show existence of such a 1-form. This is work in progress with Mike Douglas, Javier Gómez-Serrano, Yidi Qi, and Freid Tong.

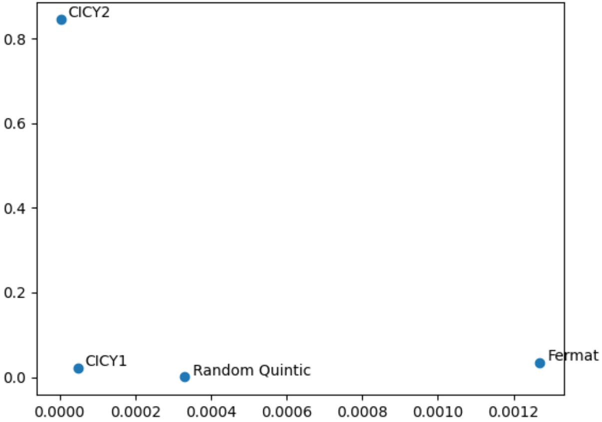
Experimental results: 1-forms and their zeros

1. **Fermat:** non-example 1; no harmonic 1-form.
2. **Random Quintic:** non-example 2; harmonic 1-form must have zeros
3. **CICY1:** large perturbation $= \frac{1}{4}$, harmonic 1-form may have zeros
4. **CICY2:** small perturbation $= \frac{1}{100}$, conjecture no zeros

y-axis: min j j

x-axis:

harmonic loss
$\frac{jjd\lambda jj_{L^1} + jjd^*\lambda jj_{L^1}}{jj\lambda jj_{L^2}}$



Monge-Ampere loss near singularities

From left to right: `CICY1`, `CICY2`, then two plots for `Quintic`

y-axis: the Monge–Ampère error $|1 \quad \omega^m/\Omega \wedge \overline{\Omega}|$

x-axis: "approximate distance to a singular locus", more precisely:

- for `CICY1` and `CICY2` as $\min\{d_2(x)/|x|^2, d_4(x)/|x|^4\}$, where $d_2(x) = \max\{|x_4^2 + x_5^2|, |x_0|^2, |x_1|^2, |x_2|^2, |x_3|^2\}$ and $d_4(x) = \max\{|x_1^4 + x_2^4 + x_3^4|, |x_0|^4, |x_4|^4, |x_5|^4\}$
- `Quintic` is smoothing of two singularities: (a) intersection of divisors $Z(f)$ and $Z(g \quad)$, (b) an ordinary double point at $a = [1 : 0 : 0 : 0 : 0]$
  third image: approximation of the distance to (a), namely $\max\{|x_0^2 + \quad + x_4^2|/|x|^2, f \quad (x)/|x|^3\}$
  fourth image: approximation of the distance to (b), namely $|x \quad (|x|, 0, 0, 0, 0)|/|x|$

Monge-Ampere loss near singularities

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  fourth image: approximation of the distance to (b), namely $|x - (|x|, 0, 0, 0, 0)|/|x|$

Formation of long necks

Quintic CICY1 CICY2

- y-axis: $\max_{v \in T_x M : |v|_{FS}=1} |v|_h$

  $|\cdot|_{FS}$: length of a tangent vector in the ambient Fubini-Study metric

  $|\cdot|_h$: length of a vector in the learned approximate Calabi-Yau metric

- x-axis: approximation of the distance to the singular locus of the initial singular variety

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