Numerical approximations of harmonic 1-forms on real loci of Calabi-Yau manifolds

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Loughborough University, 20 March 2024

Abstract: For applications in differential geometry and string theory one would like to construct Calabi-Yau manifolds of complex dimension three with the following property: it should contain a real submanifold of real dimension three that admits a harmonic nowhere vanishing 1-form. Many examples are expected to exist, but none have been proven to exist. The problem is that there is no explicit formula for the Calabi-Yau metric which makes it hard to write down the "harmonic" equation, let alone solve it. In the talk I will present numerical approximations of the Calabi-Yau metric, and numerical approximations of harmonic 1-forms, obtained by neural networks. This suggests some conjectural examples of harmonic, nowhere vanishing 1-forms. I will also show some proven non-examples, and explain the main long-term motivation for this numerical work, which is to numerically verifiably prove that there exists a genuine solution to the harmonic equation near the approximate solutions. This is work in progress, joint with Michael Douglas and Yidi Qi.

Background I: Calabi-Yau manifolds

- Calabi conjecture (Yau's theorem):
  If (Y, g, J, ω) Kähler, complex dim n with:

  $$\Omega \in \Omega^{n,0}(Y) \text{ parallel and nowhere } 0$$

  then ex. $\varphi \in C^\infty(Y)$ s.t. $\omega_{CY} = \omega + i\partial\overline{\partial}\varphi$ has $\omega_{CY}^n = \Omega \wedge \overline{\Omega}$
  ($\iff$ induced metric g_{CY} is Ricci-flat)
- Example: Fermat quintic

  $$Y := \{z = [z_0 : \cdots : z_4] \in \mathbb{CP}^4 : z_0^5 + \cdots + z_4^5 = 0\}$$

  has $\Omega \in \Omega^{n,0}(Y) \Rightarrow g_{CY}$ exists

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  $(\Leftrightarrow$ induced metric g_{CY} is Ricci-flat)
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Background II: Harmonic 1-forms on Calabi-Yaus

- Let Y be Calabi-Yau 3-fold with Calabi-Yau metric g_{CY}
- $\sigma : Y \to Y$ anti-holomorphic involution, $L := \mathrm{Fix}(\sigma)$
  example: quintic with real coefficients in $\mathbb{CP}^4$ and $\sigma([z_0 : \dots : z_4]) = [\overline{z_0} : \dots : \overline{z_4}]$
- $S^1 \times Y$ has dimension 7 and holonomy $SU(3)$. Problem: want holonomy G_2
- Define $b : S^1 \times Y \to S^1 \times Y$ as $(x, y) \mapsto (-x, \sigma(y))$
  $(S^1 \times Y)/\langle b \rangle$ $\{0, \frac{1}{2}\} \times L$ N^7

Theorem ([Joyce and Karigiannis, 2017])

If there exists $\lambda \in \mathcal{H}^1(L)$ harmonic w.r.t. $g_{CY}|_L$ that is nowhere 0, then there exists a resolution $N^7 \to (S^1 \times Y)/\langle b \rangle$ with holonomy equal to G_2.

- Goal: check if such a 1-form exists
- First Betti number → harmonic 1-forms. Nowhere 0? Must know the metric!

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Theorem ([Joyce and Karigiannis, 2017])

If there exists $\quad 2 \quad ^1(L)$ harmonic w.r.t. $g_{CY}|_L$ that is nowhere 0, then there exists a resolution $N^7 \to (S^1 \quad Y)/\langle b \rangle$ with holonomy equal to G_2.

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Non-example 1: Fermat Quintic

- $Y := \{ z = [z_0 : \quad : z_4] \in \mathbb{CP}^4 : z_0^5 + \quad + z_4^5 = 0 \}$
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$$\mathbb{RP}^3 \to L = x(\quad) = \{ x = [x_0 : \quad : x_4] \in \mathbb{RP}^4 : x_0^5 + \quad + x_4^5 = 0 \}$$

$$[x_0 : \quad : x_4] \mapsto x_0 : \quad : x_4 : \sqrt[5]{x_0^5 + \quad + x_4^5}$$

- $b^1(\mathbb{RP}^3) = 0 \Rightarrow$ no harmonic 1-form on L

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$$[x_0 : \cdots : x_4] \mapsto \left(x_0 : \cdots : x_4 : -\sqrt[5]{x_0^5 + \cdots + x_4^5} \right)$$

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$$\mathbb{R}P^3 \quad L = x(\quad) = \{ x = [x_0 : \quad : x_4] \in \mathbb{R}P^4 : x_0^5 + \quad + x_4^5 = 0 \}$$

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- $b^1(\mathbb{R}P^3) = 0 \Rightarrow$ no harmonic 1-form on L

Non-Example 2: small complex structure limit

- Another quintic in $\mathbb{CP}^4$:
  1. [Krasnov, 2009] $f = x_0(x_1^2 + x_2^2 + x_3^2 \quad x_4^2) \quad (x_1^3 + x_2^3 + x_3^3 \quad \frac{1}{2}x_4^3) \quad x_0^3$
     smoothing of ordinary double point $(1:0:0:0:0)$
     Has $Z(f_+) = \mathbb{RP}^3$, $Z(f \ ) = \mathbb{RP}^3 \# S^1 \quad S^2$
  2. $v := (x_0^2 + \quad + x_4^2)$ and $g = v \quad f$ has $Z_{\mathbb{R}}(g) = Z_{\mathbb{R}}(f \ ) \quad \mathbb{RP}^4$ so
     $: \mathbb{CP}^4 ! \ \mathbb{CP}^4, x \ 7! \ \overline{x}$ has $b^1 \quad x(\ )|_{Z(g)} \quad = 1$
  3. Take smoothing $g \ := g + \quad$, where $\quad$ generic poly
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     (even number of zeros by Poincaré{Hopf theorem and $(Z_{\mathbb{R}}(g \ )) = 0$)

sing(g) $\quad x(\quad)$

- [Tian and Yau, 1990] Calabi-Yau metrics on $Z(f \ ) \setminus \mathrm{sing}(g)$ and $Z(v) \setminus \mathrm{sing}(g)$
- [Sun and Zhang, 2019] glue these to metric on smooth $Z(g \ )$
- More non-examples from other cubics. Examples from other Fanos?

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- Another quintic in $\mathbb{CP}^4$:
  1. [Krasnov, 2009] $f \;=\; x_0(x_1^2 + x_2^2 + x_3^2 \quad x_4^2) \quad (x_1^3 + x_2^3 + x_3^3 \quad \frac{1}{2}x_4^3) \quad x_0^3$
     smoothing of ordinary double point $(1 : 0 : 0 : 0 : 0)$
     Has $Z(f_+) = \mathbb{RP}^3$, $Z(f\;\;) = \mathbb{RP}^3 \# S^1 \quad S^2$
  2. $v := (x_0^2 + \quad + x_4^2)$ and $g = v \quad f$ has $Z_{\mathbb{R}}(g) = Z_{\mathbb{R}}(f\;\;) \quad \mathbb{RP}^4$ so
     $: \mathbb{CP}^4 \,!\, \quad \mathbb{CP}^4$, $x \; 7! \; \overline{x}$ has $b^1 \quad x(\;\;)j_{Z(g)} \quad = 1$
  3. Take smoothing $g \;:=\; g + \quad$, where generic poly
  4. But: ex. incompressible S^2) [Jaco, 1980] $Z_{\mathbb{R}}(g\;) = Z_{\mathbb{R}}(g)$ is no bration over S^1
) [Tischler, 1970] no closed nowhere zero 1-form) any harmonic 1-form has zeros (even number of zeros by Poincare{Hopf theorem and $(Z_{\mathbb{R}}(g\;)) = 0$)

sing(g) $\quad x(\;\;)$

- [Tian and Yau, 1990] Calabi-Yau metrics on $Z(f\;\;) \setminus \text{sing}(g)$ and $Z(v) \setminus \text{sing}(g)$
- [Sun and Zhang, 2019] glue these to metric on smoothing $Z(g\;)$
- More non-examples from other cubics. Examples from other Fanos?

Conjectural example 3: quadric intersect quartic

- Construction of quadric intersect quartic in $\mathbb{CP}^5$, also Calabi-Yau
  1. Circle $c_a = x_1^2 + x_2^2 \quad 1$, quartic $q_a = x_3^4 + x_4^4 + x_5^4 \quad 1$
     Projectivise: $c = x_0^2 + x_1^2 + x_2^2$ and $q = x_0^4 + x_3^4 + x_4^4 + x_5^4$
  2. c and q have SO(2)-symmetry:
     $[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_0 : \cos(t)x_1 \quad \sin(t)x_2 : \sin(t)x_1 + \cos(t)x_2 : x_3 : x_4 : x_5]$
     Generic smoothings and q of c and q
  3. $\mathbb{RP}^5 \quad Z_{\mathbb{R}}(c; q) = S^1 \quad S^2$ smooth, $Z(c; q) \quad \mathbb{CP}^5$ singular
     $\mathrm{sing}(Z(c; q)) = \mathrm{sing}(c) \cap Z(q)$

     $\mathrm{sing}(Z(c; q))$

- Fantasy: $Z(c; q)$ has singular Calabi-Yau metric g_0) if q is SO(2)-invariant, then Infinitesimal SO(2)-action gives Killing field for g_0) (Ricci-flat) parallel vector field) parallel 1-form for g_0) nearly parallel 1-form for g

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- Construction of quadric intersect quartic in $\mathbb{C}P^5$, also Calabi-Yau
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Numerical Calabi-Yau metrics

- Holomorphic volume form locally $= dz^1 \wedge dz^2 \wedge dz^3$ vol $:= \wedge \bar{} \; 2 \; {}^6(Y)$
- Ample line bundle $L \,! \; Y$ and k 2 N such that L^k very ample
  Example: $Y \; CP^4$ quintic, $(O(1)j_Y)^k$
- $s_1; \ldots; s_N$ 2 $H^0(L^k)$ basis of holomorphic sections
) embeddings $= (s_1; \ldots; s_N) : Y \,! \; CP^{N-1}$
- h positive definite Hermitian form on $H^0(L^k)$ some Fubini-Study metric
  Kahler potential: $K = \log \sum_{i;j} h_{i;j} s^i \bar{s}^j$. Volume form: $!_h^3 = \mathrm{vol}_h$ 2 ${}^6(Y)$.
  If $\frac{\mathrm{vol}_h}{\mathrm{vol}} = 1$, then Ricci-flat
- [Donaldson, 2009]: choose h cleverly to minimise $\int_Y \left(\frac{\mathrm{vol}_h}{\mathrm{vol}} - 1\right)^2$

	$\frac{\mathrm{vol}_h}{\mathrm{vol}} - 1$	Comment
[Donaldson, 2009]	10^{-2}	n = 2, needs symmetries
[Headrick and Nassar, 2013]	10^{-14}	n = 3, needs symmetries
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[Douglas et al., 2022]+ours	10^{-4}	$n = 3$, quintics+complete intersections

Numerical Calabi-Yau metrics

- Holomorphic volume form locally $\Omega = dz^1 \wedge dz^2 \wedge dz^3$ $\quad \mathrm{vol}_\Omega := \Omega \wedge \bar{\Omega} \in \Omega^6(Y)$
- Ample line bundle $L \to Y$ and $k \in \mathbb{N}$ such that $L^{\otimes k}$ very ample
  Example: $Y \subset \mathbb{CP}^4$ quintic, $(\mathcal{O}(1)|_Y)^{\otimes k}$
- $s_1, \ldots, s_N \in H^0(L^{\otimes k})$ basis of holomorphic sections
  $\Rightarrow$ embedding $s = (s_1 : \ldots : s_N) : Y \to \mathbb{CP}^{N-1}$
- h positive definite Hermitian form on $H^0(L^{\otimes k})$ $\Rightarrow$ some Fubini-Study metric
  Kähler potential: $K = \log \sum_{i,j} h_{i,j} s^i \bar{s}^j$. Volume form: $\omega_h^3 = \mathrm{vol}_h \in \Omega^6(Y)$.

  If $\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} = 1$, then Ricci-flat
- [Donaldson, 2009]: choose h cleverly to minimise $\int_Y \left| \frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} - 1 \right|^2$

	$\left\| \frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} - 1 \right\|$	Comment
[Donaldson, 2009]	10^{-2}	$n = 2$, needs symmetries
[Headrick and Nassar, 2013]	10^{-14}	$n = 3$, needs symmetries
[Larfors et al., 2022]	10^{-2}	$n = 3$, not C^0, complete intersections+torics
[Douglas et al., 2022]+ours	10^{-4}	$n = 3$, quintics+complete intersections

Numerical harmonic 1-forms

Example: quintic $X := Z(f)$ $\mathbb{CP}^4$, $L := x(\quad)$ X

- $_1; \ldots; _{10}$ 2 $^1(\mathbb{RP}^4)$ closed 1-forms s.t. $T_x \mathbb{RP}^4 = (_1(x); \ldots; _{10}(x))$ 8x
- $p_1; \ldots; p_N$ polys, p_i degree d_i; for $= (_{1;1}; \ldots; _{10;N})$ 2 $\mathbb{R}^{10N}$ let

$$(x) = \sum_{\substack{i = 1; \ldots; N \\ j = 1; \ldots; 10}} {}_{i;j} \frac{p_i(x)}{jxj^{d_i}} {}_j j_L(x) \; 2 \; {}^1(L)$$

For $x_1; \ldots; x_{100000}$ 2 X nd $\min_{\text{s.t. } jj \; jj_{L^2}=1} \int_{x_1;\ldots;x_{100000}} jd \quad j + jd \quad j$

- Stone-Weierstrass) best approximations converge to harmonic form as N ! 1
- Ansatz for p_i: $A_i : \mathbb{R}^{n_i}$! $\mathbb{R}^{n_{i+1}}$ linear, $sq : \mathbb{R}^k$! $\mathbb{R}^k$ square each coordinate

$$p(x_0; \ldots; x_4) = A_k \quad sq \quad A_2 \quad sq \quad A_1(x_0; \ldots; x_4)$$

- Equivalent: neural network with activation function x 7! x^2
- Approximate metric $\frac{i}{2} @ \bar{@} K$ smooth+explicit) explicitly compute $(jd \quad j(x_i) + jd \quad j(x_i)) = \sqrt{j \; (x_1)j^2 + \quad + j \; (x_{100000})j^2}$

) minimise with tensorflow

Numerical harmonic 1-forms

Example: quintic $X := Z(f) \subset \mathbb{CP}^4$, $L := x(\;) \cap X$

- $_1, \dots, _{10} \in {}^1(\mathbb{RP}^4)$ closed 1-forms s.t. $T_x \mathbb{RP}^4 = \langle _1(x), \dots, _{10}(x) \rangle \; \forall x$
- $p_1, \dots, p_N$ polys, p_i degree d_i; for $= (_{1,1}, \dots, _{10,N}) \in \mathbb{R}^{10N}$ let

$$(x) = \sum_{\substack{i = 1, \dots, N \\ j = 1, \dots, 10}} _{i,j} \frac{p_i(x)}{|x|^{d_i}} \, _j|_L(x) \in {}^1(L)$$

Z

For $x_1, \dots, x_{100000} \in X$ find $\min_{\text{s.t. } \| \|_{L^2}=1} \sum_{x_1,\dots,x_{100000}} |d \;| + |d \;|$

- Stone-Weierstrass $\Rightarrow$ best approximations converge to harmonic form as $N \to \infty$
- Ansatz for p_i: $A_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_{i+1}}$ linear, $sq : \mathbb{R}^k \to \mathbb{R}^k$ square each coordinate

$$p(x_0, \dots, x_4) = A_k \circ \cdots \circ sq \circ A_2 \circ sq \circ A_1(x_0, \dots, x_4)$$

- Equivalent: neural network with activation function $x \mapsto x^2$
- Approximate metric $\frac{i}{2} \partial \bar{\partial}$ on X (smooth+explicit) $\Rightarrow$ explicitly compute
  $(|d \;|(x_i) + |d \;|(x_i)) = \sqrt{|\;(x_1)|^2 + \cdots + |\;(x_{100000})|^2}$
  $\Rightarrow$ minimise with tensorflow

Numerical harmonic 1-forms

Example: quintic $X := Z(f) \quad CP^4$, $L := x(\quad) \quad X$

- $_1; : : : ; \ _{10} \ 2 \quad ^1(RP^4)$ closed 1-forms s.t. $T_x RP^4 = (\ _1(x); : : : ; \ _{10}(x)) \ 8x$
- $p_1; : : : ; p_N$ polys, p_i degree d_i; for $\ = (\ _{1;1}; : : : ; \ _{10;N}) \ 2 \ R^{10N}$ let

$$(x) = \sum_{\substack{i = 1; : : : ; N \\ j = 1; : : : ; 10}} \ _{i;j} \frac{p_i(x)}{jxj^{d_i}} \ _j j_L(x) \ 2 \quad ^1(L)$$

For $x_1; : : : ; x_{100000} \ 2 \ X$ find $\min_{\substack{\text{s.t. } jj \quad jj_{L^2}=1 \\ x_1;:::;x_{100000}}} \int \quad jd \quad j + jd \quad j$

- Stone-Weierstrass $\Rightarrow$ best approximations converge to harmonic form as $N \to 1$
- Ansatz for p_i: $A_i : R^{n_i} \ ! \quad R^{n_{i+1}}$ linear, $sq : R^k \ ! \quad R^k$ square each coordinate

$$p(x_0; : : : ; x_4) = A_k \quad sq \quad A_2 \quad sq \quad A_1(x_0; : : : ; x_4)$$

- Equivalent: neural network with activation function $x \mapsto x^2$
- Approximate metric $\frac{i}{2} @ \bar{@} K$ smooth+explicit) explicitly compute $\int (jd \quad j(x_i)) + jd \quad j(x_i)) = \sqrt{j \ (x_1)j^2 + \quad + j \ (x_{100000})j^2}$
) minimise with tensorflow

Numerical harmonic 1-forms

Example: quintic $X := Z(f) \quad CP^4$, $L := x(\quad) \quad X$

- $_1; : : : ; \ _{10} \ 2 \quad ^1(RP^4)$ closed 1-forms s.t. $T_x RP^4 = (\ _1(x); : : : ; \ _{10}(x)) \ 8x$
- $p_1; : : : ; p_N$ polys, p_i degree d_i; for $= (\ _{1;1}; : : : ; \ _{10;N}) \ 2 \ R^{10N}$ let

$$(x) = \sum_{\substack{i=1;:::;N \\ j=1;:::;10}} {}_{i;j} \frac{p_i(x)}{|x|^{d_i}} \ _j|_L(x) \ 2 \quad ^1(L)$$

$$Z$$

For $x_1; : : : ; x_{100000} \ 2 \ X$ find $\min_{\text{s.t. } || \quad ||_{L^2}=1} \quad \sum_{x_1;:::;x_{100000}} |d \quad | + |d \quad |$

- Stone-Weierstrass $\Rightarrow$ best approximations converge to harmonic form as N 1
- Ansatz for p_i: $A_i : R^{n_i} \to R^{n_{i+1}}$ linear, $sq : R^k \to R^k$ square each coordinate

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) minimise with tensorflow

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- $p_1; : : : ; p_N$ polys, p_i degree d_i; for $= (\ _{1;1}; : : : ; \ _{10;N})$ 2 R^{10N} let

$$(x) = \sum_{\substack{i = 1; : : : ; N \\ j = 1; : : : ; 10}} {}_{i;j} \frac{p_i(x)}{jxj^{d_i}} \ {}_j j_L(x) \ 2 \ {}^1(L) \qquad Z$$

For $x_1; : : : ; x_{100000}$ 2 X nd $\min_{\text{s.t. } jj \quad jj_{L^2}=1} \quad {}_{x_1;:::;x_{100000}} \ jd \quad j + jd \quad j$

- Stone-Weierstrass best approximations converge to harmonic form as N ! 1
- Ansatz for p_i: $A_i : R^{n_i} ! \ R^{n_{i+1}}$ linear, sq : $R^k ! \ R^k$ square each coordinate

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) minimise with tensorflow

Numerical harmonic 1-forms

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- $p_1; : : : ; p_N$ polys, p_i degree d_i; for $= (\ _{1;1}; : : : ; \ _{10;N})\ 2\ R^{10N}$ let

$$(x) = \sum_{\substack{i = 1; : : : ; N \\ j = 1; : : : ; 10}} \ _{i;j} \frac{p_i(x)}{|x|^{d_i}} \ _j|_L(x)\ 2\ \ ^1(L)$$

$$Z$$

For $x_1; : : : ; x_{100000}\ 2\ X$ find $\min_{\text{s.t. } || \ ||_{L^2}=1} \sum_{x_1;:::;x_{100000}} |d \ | + |d \ |$

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$$p(x_0; : : : ; x_4) = A_k \quad sq \quad A_2 \quad sq \quad A_1(x_0; : : : ; x_4)$$

- Equivalent: neural network with activation function $x \mapsto x^2$
- Approximate metric $\frac{i}{2} @\bar{@}$ (X smooth+explicit) explicitly compute

$$(|d \quad |(x_i) + |d \quad |(x_i)) = \sqrt{| \ (x_1)|^2 + \quad + | \ (x_{100000})|^2}$$

$\Rightarrow$ minimise with tensorflow

Numerical harmonic 1-forms

Example: quintic $X := Z(f)$ $\mathbb{C}P^4$, $L :=$ x( ) X

- $_1; \ldots ; \,_{10}\ 2\ \ ^1(\mathbb{R}P^4)$ closed 1-forms s.t. $T_x\mathbb{R}P^4 = (\ _1(x); \ldots ; \ _{10}(x))\ 8x$
- $p_1; \ldots ; p_N$ polys, p_i degree d_i; for $= (\ _{1;1}; \ldots ; \ _{10;N})\ 2\ \mathbb{R}^{10N}$ let

$$(x) = \sum_{\substack{i = 1; \ldots ; N \\ j = 1; \ldots ; 10}} {}_{i;j} \frac{p_i(x)}{|x|^{d_i}}\ {}_j|_L(x)\ 2\ \ ^1(L)$$

$$\int$$

For $x_1; \ldots ; x_{100000}\ 2\ X$ nd

$$\min_{\substack{\text{s.t. } || \quad ||_{L^2}=1}} \sum_{x_1;\ldots;x_{100000}} |d \quad | + |d \quad |$$

- Stone-Weierstrass $\Rightarrow$ best approximations converge to harmonic form as $N!$ 1
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$$p(x_0; \ldots ; x_4) = A_k \quad \cdots \quad sq \quad A_2 \quad sq \quad A_1(x_0; \ldots ; x_4)$$

- Equivalent: neural network with activation function $x\ 7!\ x^2$
- Approximate metric $\frac{i}{2}@\bar{@}$ X smooth+explicit) explicitly compute

$$(|d \quad |(x_i) + |d \quad |(x_i)) = \frac{p}{| \ (x_1)|^2 + \quad + | \ (x_{100000})|^2}$$

) minimise with tensorflow

Numerical harmonic 1-forms

Example: quintic $X := Z(f)$ CP^4, $L :=$ x( ) X

- $_1; : : : ;\ _{10}$ 2 $^1(RP^4)$ closed 1-forms s.t. $T_x RP^4 = (\ _1(x); : : : ;\ _{10}(x))$ $8x$
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For $x_1; : : : ; x_{100000}$ 2 X nd $\min_{\text{s.t. } || \quad ||_{L^2}=1}$ $\sum_{x_1;:::;x_{100000}}$ $|d \quad | + |d \quad |$

- Stone-Weierstrass) best approximations converge to harmonic form as N ! 1
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Numerical harmonic 1-forms

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$$(x) = \sum_{\substack{i = 1; : : : ; N \\ j = 1; : : : ; 10}} \ _{i;j} \frac{p_i(x)}{|x|^{d_i}} \ _j|_L(x) \ 2 \quad ^1(L)$$

$$Z$$

For $x_1; : : : ; x_{100000} \ 2 \ X$ nd $\min_{\text{s.t. } || \quad ||_{L^2}=1} \ \sum_{x_1;:::;x_{100000}} |d \quad | + |d \quad |$

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$$(|d \quad |(x_i) + |d \quad |(x_i)) = \sqrt{| \ (x_1)|^2 + \quad + | \ (x_{100000})|^2}$$

$)$ minimise with tensorflow

Experimental results: 1-forms and their zeros

1. Fermat: non-example 1; no harmonic 1-form.
2. Random Quintic: non-example 2; harmonic 1-form must have zeros
3. CICY1: conjectural example 3; large perturbation $= \frac{1}{4}$, harmonic 1-form may have zeros
4. CICY2: conjectural example 3; small perturbation $= \frac{1}{100}$, conjecture no zeros

y-axis: $\min |\quad|$

x-axis:
harmonic loss

$$\frac{||d\quad||_{L^1} + ||d\quad||_{L^1}}{||\quad||_{L^2}}$$

Experimental results on quintic

- g = v f singular quintic from before, = 0:84x_0^5 + : : : random quintic
- Find > 0 such that g := g + has Z_R(g) di eo to Z_R(g)
  - U RP^4 nbhd of Z_R(g)
  - k := $\min_U$ jDgj > 0, M := $\min_{RP^4 nU}$ jgj > 0
  - if jj $_0$ jj$_{C^0}$ < M and jjD $_0$ jj$_{C^0}$ < k, then Z_R(g) smooth for all 0< < $_0$
  -) Z_R(g) di eo for all 0 < < $_0$ (for us $_0$ = 0:00195503)

Average of j vol$_h$ =vol 1j while j vol$_h$ =vol (x) 1j over
iteratively improving vol$_h$ max f v(x)=jj xjj ^{2} ; f (x)=jj xjj ^{3} g

Experimental results on quintic

- g = v f singular quintic from before, = $0.84x_0^5 + \dots$ random quintic
- Find > 0 such that g := g + has $Z_{\mathbb{R}}(g)$ di eo to $Z_{\mathbb{R}}(g)$
  - U $\mathbb{RP}^4$ nbhd of $Z_{\mathbb{R}}(g)$
  - $k := \min_U |Dg| > 0$, $M := \min_{\mathbb{RP}^4 \setminus U} |g| > 0$
  - if $|| \ _0 ||_{C^0} < M$ and $||D \ _0||_{C^0} < k$, then $Z_{\mathbb{R}}(g)$ smooth for all $0 < \ < \ _0$
  -) $Z_{\mathbb{R}}(g)$ di eo for all $0 < \ < \ _0$ (for us $_0 = 0.00195503$)

Average of $|\mathrm{vol}_h/\mathrm{vol} \ \ 1|$ while iteratively improving vol_h

$|\mathrm{vol}_h/\mathrm{vol} \ (x) \ \ 1|$ over max$\{v(x)=||x||^2 ; f \ (x)=||x||^3\}$

Experimental results on quintic

- $g = v \quad f$ singular quintic from before, $= 0{:}84x_0^5 + :::$ random quintic
- Find > 0 such that $g := g +$ has $Z_R(g\ )$ di eo to $Z_R(g)$
  - $U \quad RP^4$ nbhd of $Z_R(g)$
  - $k := \min_U jDgj > 0$, $M := \min_{RP^4 nU} jgj > 0$
  - if $jj\ _0\ jj_{C^0} < M$ and $jjD\ _0\ jj_{C^0} < k$, then $Z_R(g\ )$ smooth for all $0 < \ < \ _0$
  - $) \quad Z_R(g\ )$ di eo for all $0 < \ < \ _0$ (for us $_0 = 0{:}00195503$)

Average of $j\ vol_h = vol \quad 1j$ while $j\ vol_h = vol\ (x) \quad 1j$ over

iteratively improving vol_h $\max f\ v(x) = jjxjj^2 ; f\ (x) = jjxjj^3 g$

Experimental results on quintic

Neck formation

$\max_{v \in T_x, \|v\|_{FS}=1} \|v\|_h$ over $\max\{v(x) = \|x\|^2, f(x) = \|x\|^3\}$

1-form has even number of zeros

k-medoid clustering loss of 500 points with smallest $|\omega|(x)$ over number of clusters (heuristic: "elbow" $k = 4$ is optimal number of clusters

Experimental results on quadratic\quartic

$= \frac{1}{4}$ $= \frac{1}{100}$

" Training loss

Loss over distance from singularity

Experimental results on quadratic\quartic

$= \frac{1}{4}$

$= \frac{1}{100}$

" Metric stretching over distance from singularity

Experimental results on quadric\quartic

$c = \quad x_0^2 + x_1^2 + x_2^2$ and $q = \quad x_0^4 + x_3^4 + x_4^4 + x_5^4$

Set $x_0 = 1$ and $x_3 = x_4 = 0$ y f $(x_1; x_2)$ 2 R^2 : $x_1^2 + x_2^2 = 1$ g f 1g

1-form restricted to this

Bonus motivation

Proposition

For all > 0 there exists > 0 such that

$$\left\| \frac{\mathrm{vol}_h}{\mathrm{vol}} - 1 \right\|_{L^p_1} < \quad \Rightarrow \quad \| g_{\mathrm{approx}} - g_{CY} \|_{L^p_1} < \,.$$

Proposition

For all > 0 there exists > 0 such that the following is true: for $\in \Omega^1(L^3)$ such that $\Delta_{\mathrm{approx}} = 0$ and $\| \|_{L^2; g_{\mathrm{approx}}} = 1$ and $\min_L | | >$ let $\tilde{} \in [\ ]$ be the unique Δ_{CY}-harmonic 1-form. Then:

$$\| g_{\mathrm{approx}} - g_{CY} \|_{L^p_1} < \quad \Rightarrow \quad | \tilde{} - |(x) < \frac{}{2} \Rightarrow | \tilde{} |(x) > \frac{}{2} \text{ for all } x \in L.$$

- Find: g_{approx} with $\left\| \frac{\mathrm{vol}_h}{\mathrm{vol}} - 1 \right\|_{L^p_1} <$, with $\Delta_{\mathrm{approx}} = 0$ and $\min_L | | >$
- $\Rightarrow$ there exists nowhere vanishing g_{CY}-harmonic 1-form on L.

Bonus motivation

Proposition

For all $>$ 0 there exists $>$ 0 such that

$$\frac{vol_h}{vol} \quad 1 \quad {}_{L^p_1} < \quad) \;\; \| \;\; g_{approx} \quad g_{CY} \|_{L^p_1} < \; :$$

Proposition

For all $>$ 0 there exists $>$ 0 such that the following is true:
for $2 \quad ^1(L^3)$ such that $_{approx} = 0$ and $\| \quad \|_{L^2; g_{approx}} = 1$ and $\min_L | \quad | >$ let $\tilde{} \; 2 \; [\quad]$ be the unique $_{CY}$-harmonic 1-form. Then:

$$\| g_{approx} \quad g_{CY} \|_{L^p_1} < \quad) \;\; | \quad \tilde{} \quad |(x) < \frac{}{2} \;) \; | \tilde{} |(x) > \frac{}{2} \text{ for all } x \in L:$$

- Find: g_{approx} with $\frac{vol_h}{vol} \quad 1 \quad {}_{L^p_1} <$, with $_{approx} = 0$ and $\min_L | \quad | >$
-) there exists nowhere vanishing g_{CY}-harmonic 1-form on L

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Proposition

For all > 0 there exists > 0 such that

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Proposition

For all > 0 there exists > 0 such that the following is true:
for $\in {}^1(L^3)$ such that $_{approx} = 0$ and $\| \|_{L^2; g_{approx}} = 1$ and $\min_L | | >$ let $e \in [\;]$ be the unique $_{CY}$-harmonic 1-form. Then:

$$\| g_{approx} - g_{CY} \|_{L^p_1} < \;\Rightarrow\; | e - (x) | < \frac{}{2} \;\Rightarrow\; | e(x) | > \frac{}{2} \text{ for all } x \in L.$$

- Find: g_{approx} with $\left\| \frac{\mathrm{vol}_h}{\mathrm{vol}} - 1 \right\|_{L^p_1} <$, with $_{approx} = 0$ and $\min_L | | >$
- $\Rightarrow$ there exists nowhere vanishing g_{CY}-harmonic 1-form on L

Thank you for the attention!

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