An example of a G2-instanton on a resolution of $(K3 \times T^3)/\mathbb{Z}_2^2$ coming from a stable bundle

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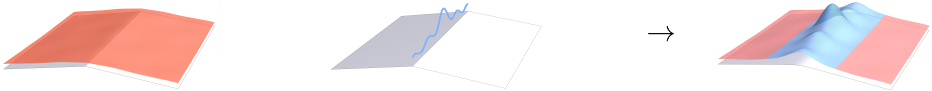
Spinorial and Octonionic Aspects of G_2 and Spin(7) Geometry, Banff



Abstract: I will begin with a brief explanation of what G_2-instantons and G_2-manifolds are. There is a general construction by Joyce-Karigiannis for G_2-manifolds. Ignoring all analysis, I will explain one example of their construction. The example is the resolution of $(K3 \times T^3)/\mathbb{Z}_2^2$ for a very explicit K3 surface. Furthermore, there is a construction method for G_2-instantons on Joyce-Karigiannis manifolds. I will explain the ingredients needed for the construction, say nothing about the proof, and then explain one example of the ingredients.

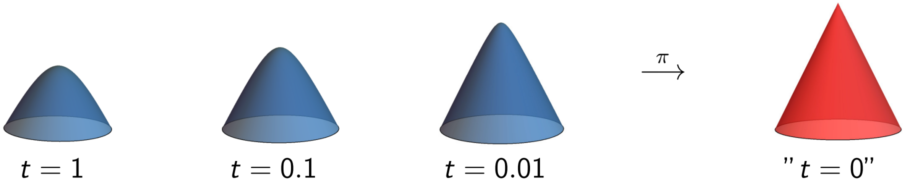
Hyperkähler 4-manifolds

1. (y_0, y_1, y_2, y_3) coordinates on $\mathbb{C}^2$, Hyperkähler triple

$$\omega_1 = dy_0 \wedge dy_1 + dy_2 \wedge dy_3, \qquad \omega_2 = dy_0 \wedge dy_2 + dy_3 \wedge dy_1,$$
$$\omega_3 = dy_0 \wedge dy_3 + dy_1 \wedge dy_2$$

   Invariant under $(-1) : \mathbb{C}^2 \to \mathbb{C}^2, \quad x \mapsto -x \Rightarrow \omega_1, \omega_2, \omega_3 \in \Omega^2(\mathbb{C}^2/\{\pm 1\})$

2. Blowup $\Rightarrow$ complex manifold X_{EH} with $\pi : X_{EH} \to \mathbb{C}^2/\{\pm 1\}$
   Eguchi-Hanson $\Rightarrow$ ex. $\widetilde{\omega}_1^{(t)}, \widetilde{\omega}_2^{(t)}, \widetilde{\omega}_3^{(t)} \in \Omega^2(X)$ Hyperkähler triple



The exceptional Lie group G_2

- On $\mathbb{R}^7 = \overset{(x_1,x_2,x_3)}{\mathbb{R}^3} \times \overset{(y_0,y_1,y_2,y_3)}{\mathbb{C}^2}$

$$\varphi_0 := dx_1 \wedge dx_2 \wedge dx_3 - \sum_{i=1}^{3} dx_i \wedge \omega_i, \quad *\varphi_0 = \mathrm{vol}_{\mathbb{C}^2} - \sum_{\substack{(1,2,3),(2,3,1),\\(3,1,2)}} dx_i \wedge dx_j \wedge \omega_k$$

- $G_2 := \mathrm{Stab}_{GL(7,\mathbb{R})}(\varphi_0)$. Remark: $\varphi_0(u,v,w) = \langle u \times v, w \rangle$ where $u, v, w \in \mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$
- $\varphi \in \Omega^3(M^7)$ is G_2-structure if: for all $x \in M$ exists $F : T_xM \to \mathbb{R}^7$ s.t. $F^*\varphi_0 = \varphi(x)$
- Fact: $G_2 \subset SO(7)$ therefore φ induces metric g_φ and $*_\varphi$

Theorem (Fernández-Gray '82)

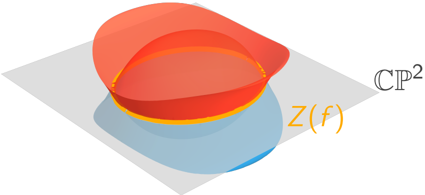
$\mathrm{Hol}(g_\varphi) \subset G_2$ *if and only if* $d\varphi = 0$ *and* $d(*_\varphi \varphi) = 0$.

- Which 7-manifolds admit holonomy G_2 metrics? Difficult!

Example of a G_2-manifold

- $f \in \mathbb{C}[x,y,z]$ homogeneous degree 6 polynomial

$$Y := \{(x:y:z:w) \in \mathbb{CP}(1,1,1,3) : f(x,y,z) = w^2\}$$



K3 surface with Hyperkähler triple $\omega_1, \omega_2, \omega_3$

On $T^3 \times Y$:

$$\varphi_{T^3\times Y} = dx_1 \wedge dx_2 \wedge dx_3 - \sum dx_i \wedge \omega_i$$

$$\alpha : Y \to Y \qquad (x:y:z:w) \mapsto (x:y:z:-w)$$

$$\sigma : Y \to Y \qquad (x:y:z:w) \mapsto (\overline{x}:\overline{y}:\overline{z}:\overline{w})$$

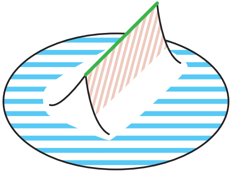
- Involutions, $\alpha^*(\omega_1) = \omega_1, \quad \alpha^*\omega_2 = -\omega_2, \quad \alpha^*\omega_3 = -\omega_3$
  $\sigma^*(\omega_1) = -\omega_1, \quad \sigma^*\omega_2 = \omega_2, \quad \sigma^*\omega_3 = -\omega_3$ extend to $T^3 \times Y$:

$$\widetilde{\alpha} : T^3 \times Y \to T^3 \times Y \qquad (x_1,x_2,x_3),p \mapsto (x_1, \frac{1}{2} - x_2, -x_3), \alpha(p)$$

$$\widetilde{\sigma} : T^3 \times Y \to T^3 \times Y \qquad (x_1,x_2,x_3),p \mapsto (-x_1, x_2, \frac{1}{2} - x_3), \sigma(p)$$

- $\widetilde{\alpha}^*\varphi_{T^3\times Y} = \varphi_{T^3\times Y}$ and $\widetilde{\sigma}^*\varphi_{T^3\times Y} = \varphi_{T^3\times Y} \rightsquigarrow \varphi_{orbi} \in \Omega^3((T^3 \times Y)/\langle\widetilde{\alpha},\widetilde{\sigma}\rangle)$

Example of a G_2-manifold (continued)



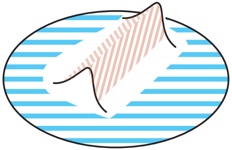
$(T^3 \times Y)/\langle\widetilde{\alpha}, \widetilde{\sigma}\rangle$

Singular set $\mathrm{fix}(\widetilde{\sigma})$, pointwise orthonormal basis e_1, e_2, e_3

Neighbourhood $U \cong \mathrm{fix}(\widetilde{\sigma}) \times (B^4/\{\pm 1\})$

$\varphi_U := e_1 \wedge e_2 \wedge e_3 - \sum e_i \wedge \omega_i$, for $\omega_i \in \Omega^2(\mathbb{C}^2/\{\pm 1\})$

Far away from singular set φ_{orbi}



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$\pi : X_{EH} \to \mathbb{C}^2/\{\pm 1\}$, nbhd $\widetilde{U} \cong \mathrm{fix}(\widetilde{\sigma}) \times \pi^{-1}(B^4/\{\pm 1\})$

$\varphi_{\widetilde{U}} := e_1 \wedge e_2 \wedge e_3 - \sum e_i \wedge \omega_i^{(t)}$, for $\omega_i^{(t)} \in \Omega^2(X_{EH})$

As before φ_{orbi}

Theorem (Joyce-Karigiannis '21)

On M exists a G_2-structure $\widetilde{\varphi}^{(t)}$ such that $\nabla\widetilde{\varphi}^{(t)} = 0$ and

$$\left|\widetilde{\varphi}^{(t)} - \varphi_{orbi}\right| \text{ small far away from } \mathrm{fix}\,\widetilde{\sigma} \text{ and } \left|\widetilde{\varphi}^{(t)} - \varphi_{\widetilde{U}}\right| \text{ small near } \mathrm{fix}\,\widetilde{\sigma}.$$

G_2-instantons

- Connection A on bundle over $(Y^4, \omega_1, \omega_2, \omega_3)$ is anti-self-dual instanton if

$$F_A \wedge \omega_i = 0 \text{ for } i = 1, 2, 3$$

- Connection A on bundle over $(M^7, \varphi, \psi = *\varphi)$ is G_2-instanton if $F_A \wedge \psi = 0$
- Example on $p : T^3 \times Y \to Y$: A anti-self-dual instanton over Y

$$F_{p^*A} \wedge \psi = F_{p^*A} \wedge \left(\mathrm{vol}_Y - \sum_{\substack{(1,2,3),(2,3,1),\\(3,1,2)}} \mathrm{d}x_{ij} \wedge \omega^k \right)$$

$$= \underbrace{F_{p^*A} \wedge \mathrm{vol}_Y}_{\text{6-form on 4-fold is 0}} - \sum_{\substack{(1,2,3),(2,3,1),\\(3,1,2)}} \mathrm{d}x_{ij} \wedge \underbrace{F_{p^*A} \wedge \omega_k}_{=0 \text{ because anti-self-dual}} = 0$$

$\Rightarrow p^*A$ is G_2-instanton

A G_2-instanton on $(T^3 \times Y)/\langle\widetilde{\alpha}, \widetilde{\sigma}\rangle$

- On $\mathbb{CP}^2$ write $E = T\mathbb{CP}^2$, have

$$\Omega^2_{\mathbb{C}} = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$$

∇^{LC} on $\mathbb{CP}^2$ is Hermite-Einstein:

$$F^{2,0} = 0, F^{0,2} = 0, \langle F^{1,1}, \omega\rangle = \lambda \, \mathrm{Id} \in \Omega^0(\mathbb{CP}^2, \mathfrak{u}(E)) \text{ for } \lambda \in \mathbb{C}$$

$\Leftrightarrow$ E is stable (Donaldson-Uhlenbeck-Yau theorem)
$\Rightarrow$ for $\rho : Y \to \mathbb{CP}^2$ have that $\rho^* E$ is stable
$\Leftrightarrow$ $\rho^* E$ admits Hermite-Einstein connection A

- $\lambda : \mathrm{U}(2) \to \mathrm{PU}(2) = \mathrm{SO}(3) \rightsquigarrow$ extend $U(2)$-connection A to PU(2)-connection $\widetilde{A}$

$$F^{2,0}_{\widetilde{A}} = 0, F^{0,2}_{\widetilde{A}} = 0, \langle F^{1,1}_{\widetilde{A}}, \omega\rangle = [0] \in \Omega^0_{\mathbb{C}}(Y, \mathfrak{pu}(\rho^* E)), \text{ because } [\mathrm{Id}] = [0] \in \mathfrak{pu}(2)$$

$(\Omega^2_+)_{\mathbb{C}} = \Omega^{2,0} \oplus \Omega^{0,2} \oplus \langle\omega\rangle$, so $F_{\widetilde{A}}$ is anti-self-dual

- $\sigma' := d\sigma : \rho^* E \to \rho^* E$ lift of σ preserves $\widetilde{A}$, analog lift α' for α; extend to $T^3 \times Y$
- $\Rightarrow p^* \widetilde{A}$ is G_2-instanton on $T^3 \times Y$, descends to $(p^* \rho^* E)/\langle\alpha', \sigma'\rangle$

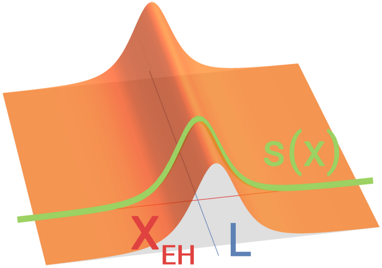
A G_2-instanton on $\widetilde{U} \cong L^3 \times X_{EH}$

- Moduli space

$$\mathcal{M}(X_{EH}) = \{\text{framed anti-self-dual connections on } X_{EH}\}/\{\text{bundle isomorphism}\}$$

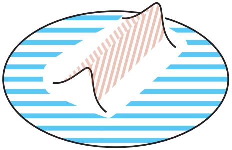
- X_{EH} Hyperkähler $\Rightarrow \mathcal{M}(X_{EH})$ Hyperkähler I_1, I_2, I_3
- $s : L^3 \to \mathcal{M}(X_{EH})$, local frame (x_1, x_2, x_3) on L. Fueter section $\Leftrightarrow$

$$0 = \sum_{i=1}^{3} I_i \left(\mathrm{d}s \left(\frac{\partial}{\partial x_i} \right) \right)$$



- $s \rightsquigarrow s(A)$ connection over $L \times X_{EH}$
- s Fueter section $\Rightarrow s(A)$ is close to being G_2-instanton
- Example: $s(x) := A$ for all x, constant Fueter section

A glued G_2-instanton



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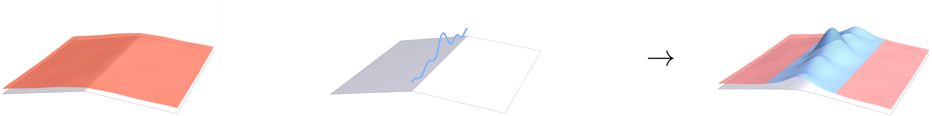
$s(A)$ close to being G_2-instanton
on $\mathrm{fix}(\widetilde{\sigma}) \times \pi^{-1}(B^4/\{\pm 1\})$
(coming from Fueter section)
$p^*\widetilde{A}$ is G_2-instanton on $(T^3 \times Y)/\langle \widetilde{\alpha}, \widetilde{\sigma} \rangle$
(coming from stable bundle)

Theorem (P. '22)

On M exists a G_2-instanton $\widetilde{A}_t$

$$\left|\widetilde{A}_t - p^*\widetilde{A}\right| \text{ small far away from } \mathrm{fix}\,\widetilde{\sigma} \text{ and } \left|\widetilde{A}_t - s(A)\right| \text{ small near } \mathrm{fix}\,\widetilde{\sigma}.$$



- Needs s rigid, no non-constant example known

Thank you for the attention!

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