# A Numerically Verifiable Proof for M-theory Compactifications*

Michael Douglas[1], Daniel Platt[2], Yidi Qi[3]

[1]Harvard University, Stony Brook University [2]King's College London [3]Northeastern University

## Motivation

$Y$ **Calabi-Yau 3-fold** (real dimension $6$) with involution $\sigma$

Theorem (Joyce-Karigiannis [1]):
If $fix(\sigma)$ admits a **nowhere vanishing harmonic vector field**, then $(Y \times S^1)/\langle\sigma\rangle$ admits a metric with holonomy $G_2$.

These manifolds are needed in M-theory (a flavour of string theory)

Problem: **only conjectural examples** are known

Goal: prove one of the conjectural examples works!

## Background

Calabi-Yau manifold $Y$ in $\mathbb{CP}^5$ be (a small perturbation of)

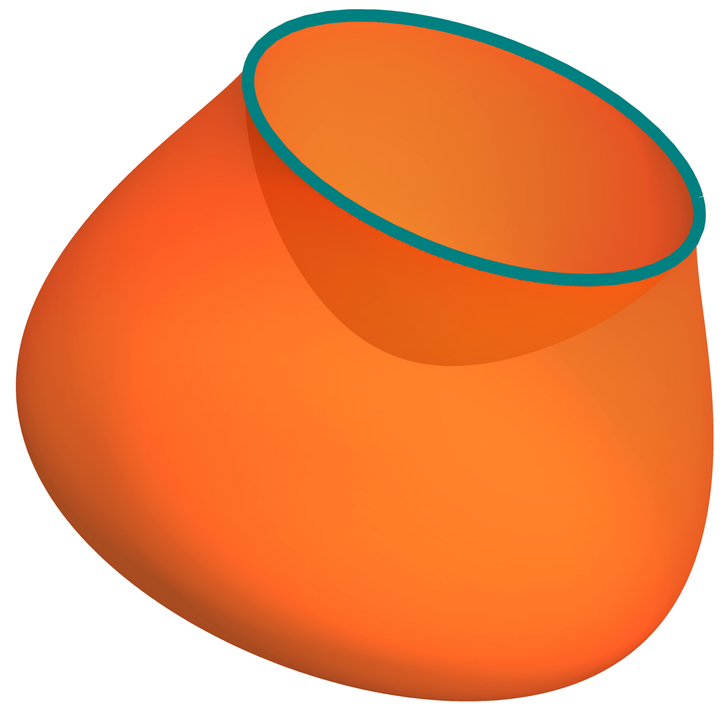
$$Y_0 := \{\vec{z} \in \mathbb{CP}^5 : p(\vec{z}) = 0 \text{ and } q(\vec{z}) = 0\}$$

where $p$ is degree $2$ polynomial, $q$ is degree $4$ polynomial.
$\sigma : Y \to Y$ the conjugation map, then "real locus" is

$$Y_{\mathbb{R}} := fix(\sigma) = \{\vec{z} = (z_0 : \cdots : z_6) \in Y : z_0, \ldots, z_6 \in \mathbb{R}\}$$

$Y$ has two metrics:

| $g_{CY}$ | $g_{approx}$ (see [2]) |
|---|---|
| Calabi-Yau metric, has $Ricci(g_{CY}) = 0$, no closed form expression known | An approximation of $g_{CY}$, has $Ricci(g_{approx}) \neq 0$ **but small**, very explicit formulae given |



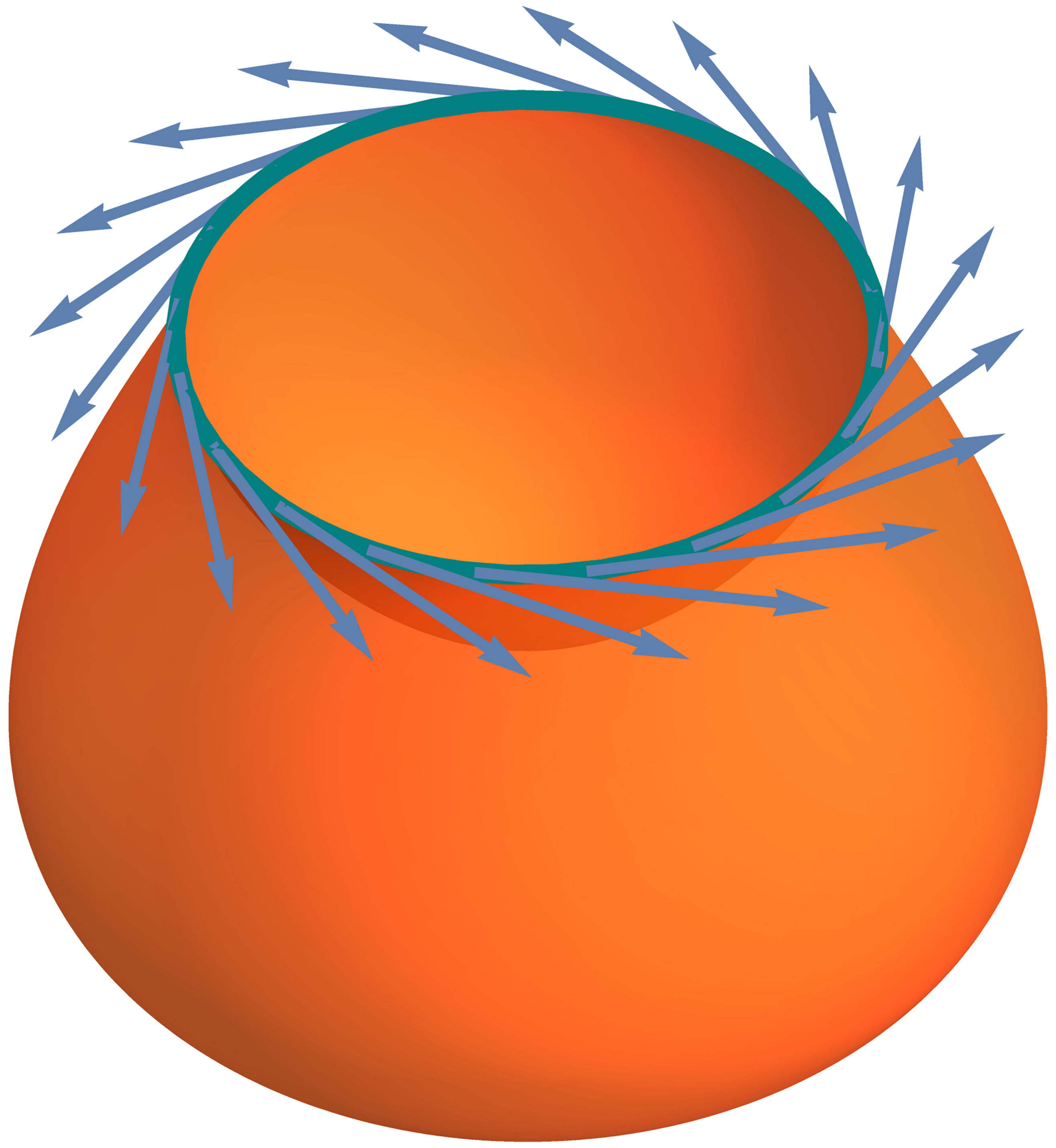
$Y$ in orange,
$Y_{\mathbb{R}}$ in green

## Question

**Does $Y_{\mathbb{R}}$ admit a harmonic vector field with no zeros?**

$$V \in \Gamma(TY_{\mathbb{R}}), \quad \Delta_{CY}(V) = 0 \quad ?$$

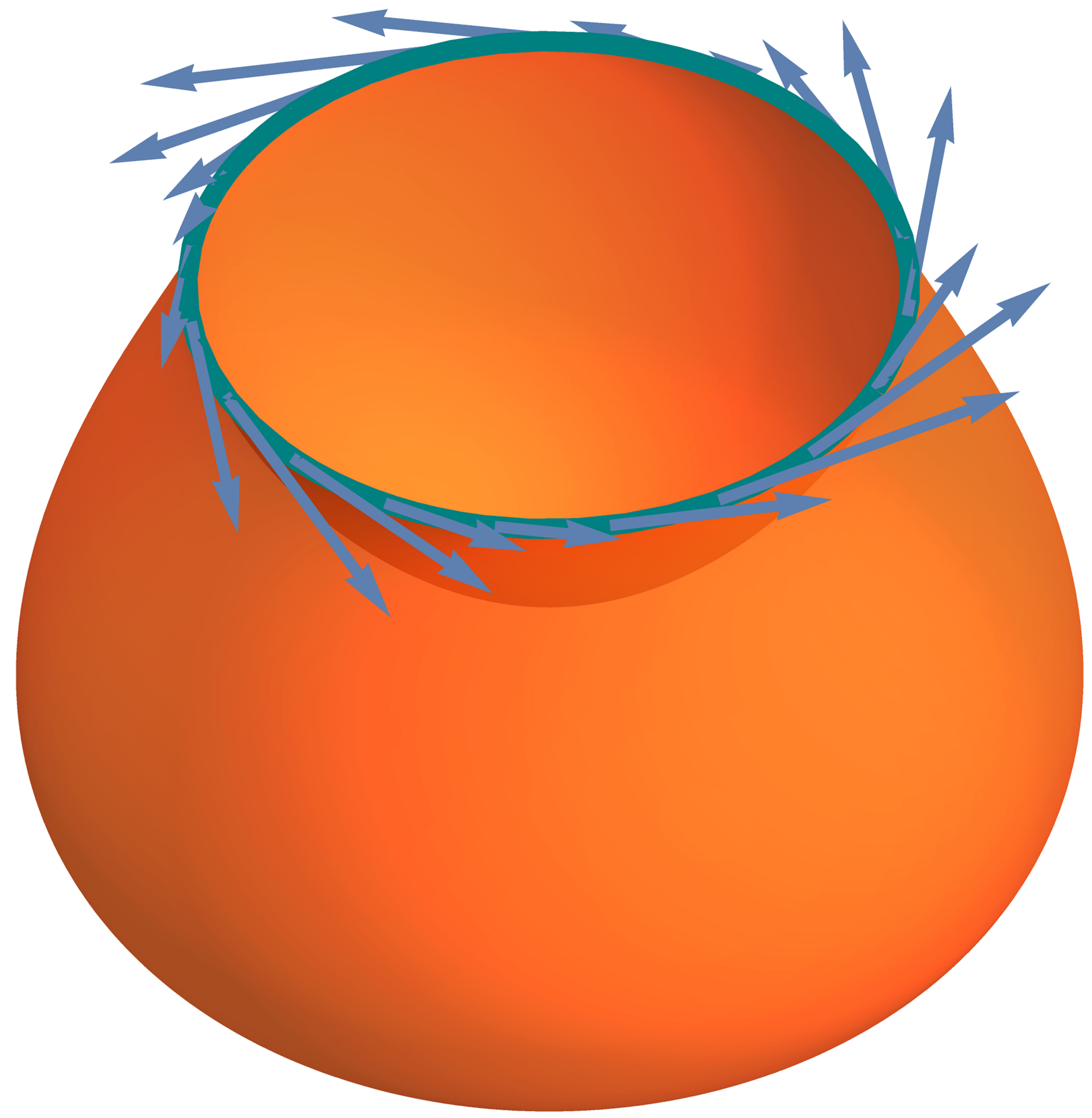
Problem: $\Delta_{CY}$ depends on the non-explicit metric $g_{CY}$



A vector field $\tilde{V}$ on $Y_{\mathbb{R}}$ which is harmonic with respect to $g_{approx}$. It can be computed explicitly and is nowhere $0$, with a good bound from below: $|\tilde{V}| > \epsilon$.

## Our approach

1. Find harmonic (unit norm) vector field $\tilde{V} \in \Gamma(TY_{\mathbb{R}})$ w.r.t. $g_{approx}$ with
$$|\tilde{V}| > \epsilon \quad \textbf{everywhere.}$$
2. ("Stability estimate") Find $C > 0$ with
$$||g_{CY} - g_{approx}|| \leq C \cdot ||Ricci(g_{approx})||$$
3. Classical fact (Hodge theory): ex. $V \in \Gamma(TY_{\mathbb{R}})$ that is $g_{CY}$-harmonic in same cohomology class as $\tilde{V}$
4. Find $D > 0$ with
$$||V - \tilde{V}|| \leq D \cdot ||g_{CY} - g_{approx}||$$
5. Conclusion: if
$$||Ric(g_{approx})|| < \epsilon \cdot \frac{1}{C} \cdot \frac{1}{D}$$
then
$$|V| > |\tilde{V}| - |V - \tilde{V}| > \epsilon - \epsilon\frac{1}{C} \cdot \frac{1}{D} \cdot C \cdot D = 0$$
**$V$ has no zeros!**



The perturbed vector field $V$ on $Y_{\mathbb{R}}$ which is harmonic with respect to $g_{approx}$. If $||Ricci(g_{approx})||$ is very small, then the metrics $g_{CY}$ and $g_{approx}$ are so close to each other that harmonic vector fields of the two metrics are almost the same. Because $\tilde{V}$ was nowhere $0$, the new $V$ is nowhere $0$, even though the bound from below may be worse.

## References

[1] Dominic Joyce and Spiro Karigiannis. A new construction of compact torsion-free $G_2$-manifolds by gluing families of Eguchi-Hanson spaces. *J. Differential Geom.*, 117(2):255–343, 2021.
[2] Michael Douglas, Subramanian Lakshminarasimhan, and Yidi Qi. Numerical calabi-yau metrics from holomorphic networks. In *Mathematical and Scientific Machine Learning*, pages 223–252. PMLR, 2022.

*This work is supported by the Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics