The Seiberg-Witten Equations

Talk for the Geometric Analysis Seminar on 1 May 2019

In the beginning I will recall the Seiberg-Witten equations. Following this, I will explain why the moduli space of solutions to the SW-equations is compact. Afterwards I will explain the canonical Spin^c structure and Dirac operator on Kähler 4-folds, and use this to compute their SW-invariants. The talk will end with a nod to the relation between SW-invariants and Gromov-Witten invariants discovered by Taubes.

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1 Prerequisites

1.1 Linear Algebra

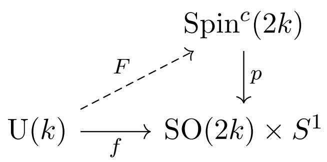
Definition 1.1. For $n > 2$, $\mathrm{Spin}(n)$ is defined to be the universal cover of $\mathrm{SO}(n)$, $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ and

$$\mathrm{Spin}^c(n) := \frac{\mathrm{Spin}(n) \times S^1}{\{\pm 1\}}. \tag{1}$$

Proposition 1.2. *Define*

$$\begin{aligned} f : \mathrm{U}(k) &\to \mathrm{SO}(2k) \times S^1 & \qquad p : \mathrm{Spin}^c(2k) &\to \mathrm{SO}(2k) \times S^1 \\ A &\mapsto (A, \det A), & [g, z] &\mapsto (\lambda g, z^2). \end{aligned} \tag{2}$$

Then there exists a lift $F : \mathrm{U}(k) \to \mathrm{Spin}^c(2k)$ *of* f *making the following diagram commute:*



Proof. [Fri00, p.27] ☐

$\mathrm{Spin}(2k)$ has two irreducible representations that are *not* pulled back via λ from representations of $\mathrm{SO}(2k)$. These are called *Spin representations* and denoted as Δ_{2k}^+ and Δ_{2k}^-, i.e.:

$$\varrho : \mathrm{Spin}(2k) \curvearrowright \Delta_{2k} := \Delta_{2k}^+ \oplus \Delta_{2k}^- \tag{3}$$

For $e_i \in \mathbb{R}^n$ we have Clifford multiplication $e_i : \Delta_{2k}^{\pm} \to \Delta_{2k}^{\mp}$.

Definition 1.3. We define the $\mathrm{Spin}^c(2k)$ representation

$$\begin{aligned} \varrho^c : \mathrm{Spin}^c(2k) &\to \mathrm{Aut}(\Delta_{2k}) \\ [g, z] &\mapsto z \cdot \varrho(g) \end{aligned} \tag{4}$$

which decomposes into two representations $\varrho^c_{\pm} : \mathrm{Spin}^c(2k) \to \mathrm{Aut}(\Delta_{2k}^{\pm})$.

Proposition 1.4. *For* $\Phi \in \Delta_4^+$ *define* $\omega^\Phi \in \Lambda^2(\mathbb{R}^4, \mathbb{C})$ *by*

$$\omega^\Phi(X, Y) = \langle X \cdot Y \cdot \Phi, \Phi \rangle + \langle X, Y \rangle |\Phi|^2. \tag{5}$$

Then

1. $\omega^\Phi \in \Lambda^2_+(\mathbb{R}^4, i\mathbb{R})$, *and*

2. $\langle \omega^\Phi \cdot \Phi, \Phi \rangle = -2\,|\Phi|^4$.

Proof. [Fri00, p.133] □

1.2 On Manifolds

Proposition 1.5. *Let J be an almost complex structure on M^{2k}. Then we have a canonical* Spin^c*-structure.*

Proof. The almost complex structure defines a $\mathrm{U}(k)$-reduction Q of the Frame bundle $\mathrm{GL}(M)$:

$$\begin{array}{ccc} \mathrm{U}(k) & \longrightarrow & Q \\ & & \downarrow \\ & & M \end{array}$$

Using the map $F : \mathrm{U}(k) \to \mathrm{Spin}^c(2k)$ from Proposition 1.2 we can define:

$$P := Q \times_{\mathrm{U}(k)} \mathrm{Spin}^c(2k), \tag{6}$$

the canonical Spin^c-structure. □

Definition 1.6. Let

$$S^+ := P \times_{\mathrm{Spin}^c(4)} \Delta_4^+, \qquad S^- := P \times_{\mathrm{Spin}^c(4)} \Delta_4^-, \tag{7}$$

be the *positive and negative Spinor bundles.*

Definition 1.7. The Clifford multiplication on $\Delta_4^\pm$ gives rise to a Clifford multiplication $TM \otimes S^\pm \to S^\mp$. For $\Phi \in \Gamma(S^+)$ define $\omega^\Phi \in \Omega^2_+(M, i\mathbb{R})$ as before.

Definition 1.8. We have a representation

$$\begin{aligned} \mathrm{Spin}^c(n) \times \mathbb{C} &\to \mathbb{C} \\ [g, z] \cdot x &\mapsto z^2 x \end{aligned} \tag{8}$$

and use this to define the *determinant bundle*

$$L := P \times_{\mathrm{Spin}^c(n)} \mathbb{C}. \tag{9}$$

Proposition 1.9. *On an almost complex manifold (X, J) we have*

$$L \simeq K_X := \Lambda^k(E) \tag{10}$$

for $E := Q \times_{\mathrm{U}(k)} \mathbb{C}^k$.

Proof. [Fri00, p.52, Example 2] □

The Levi-Civita connection $\nabla^{\mathrm{LC}} \in \mathcal{A}(\mathrm{SO}(M))$ and any choice $A \in \mathcal{A}(L)$ give rise to a coupled Dirac operator $D^A : \Gamma(S^+) \to \Gamma(S^)$.

Definition 1.10. The unperturbed Seiberg-Witten equations for a pair (Φ, A) with $\Phi \in \Gamma(S^+)$ and $A \in \mathcal{A}(L)$ are

$$\begin{cases} D^A\Phi & = 0 \\ F_A^+ & = -\frac{1}{4}\omega^\Phi \end{cases} \tag{SW}$$

2 Compactness of the Moduli Space of Solutions to the Seiberg-Witten Equations

Remember the Lichnerowicz formula (Weitzenböck identity for the Dirac operator):

$$D^A D^A \Phi = (\nabla^A)^* \nabla^A \Phi + \frac{\mathrm{scal}}{4}\Phi + \frac{1}{2} F_A^+ \cdot \Phi. \tag{11}$$

Proposition 2.1 (Lemma 2 in [KM94])**.** *If* (Φ, A) *is a solution of the SW-equations over a compact manifold* (M^4, g)*, then*

$$|\Phi(x)|^2 \le -\min_{y\in M} \mathrm{scal}(y) \tag{12}$$

or Φ *vanishes everywhere.*

Proof. If $|\Phi|^2$ attains its maximum in $x \in M$, then $\Delta |\Phi|^2 (x) \ge 0$. Let (e_1, e_2, e_3, e_4) be an orthornormal basis of TM around x, such that $\mathrm{div}(e_i) = 0$. Then, at x:

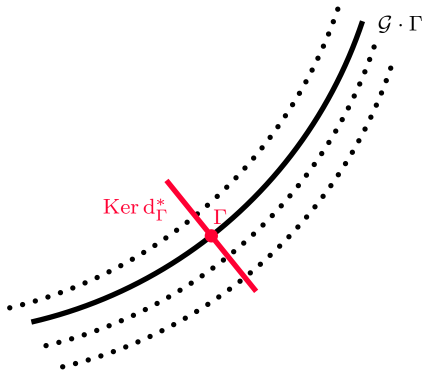
$$\begin{aligned} 0 &\le -\sum_{i=1}^4 \nabla_i \nabla_i \langle \Phi, \Phi\rangle \\ &= 2\langle (\nabla^A)^* \nabla^A \Phi, \Phi\rangle - 2\langle \nabla^A \Phi, \nabla^A \Phi\rangle \\ &\le 2\langle (\nabla^A)^* \nabla^A \Phi, \Phi\rangle \\ &= -\frac{\mathrm{scal}}{2} |\Phi|^2 - \langle \frac{1}{4}\omega^\Phi \Phi, \Phi\rangle \\ &= -\frac{\mathrm{scal}}{2} |\Phi|^2 - \frac{1}{2} |\Phi|^4, \end{aligned}$$

where we used the Lichnerowicz formula and the first SW-equation which asserts that $D^A\Phi = 0$. This proves the claim. □

Proposition 2.2. *The linearisation of the SW-equations (SW) modulo transformations of the gauge group* $\mathcal{G}$ *of* L *is elliptic.*

Proof. Fix $\Gamma \in \mathcal{A}(L)$. We then can write any connection $A \in \mathcal{A}(L)$ as $A = \Gamma + B$ for $B \in \Omega^1(M, \mathrm{End}(L)) = \Omega^1(M, \mathbb{C})$.

The space $\mathrm{Ker}\, \mathrm{d}_\Gamma^* \subset \Omega^1(M)$ is transverse to the orbit $\mathcal{G} \cdot \Gamma$. A proof of this fact can be found in [DK90, Section 2.3.1].



The SW-equations modulo gauge action are therefore equations on pairs (Φ, B) such that

$$\begin{cases} D^{\Gamma+B}\Phi &= 0 \\ F_{\Gamma+B}^+ &= -\frac{1}{4}\omega^\Phi \\ \mathrm{d}_\Gamma^* B &= 0 \end{cases} \tag{13}$$

which, at a point (Φ, B) has the linearisation acting on (Ψ, C) given by

$$\begin{cases} D^A\Psi + C \cdot \Phi &= 0 \\ \mathrm{d}_A^+ C &= \text{(zero-th order terms)} \\ \mathrm{d}_A^* C &= 0. \end{cases} \tag{14}$$

The first of these equations is elliptic, because $D^A D^A$ has the same symbol as the Laplacian by the Lichnerowicz formula.

The second two of these equations are elliptic which can be seen from the Weitzenböck formula for the operator $(d^+ \oplus d^*)$ (see [FU91, Formula 6.25]). ☐

Proposition 2.3. *The space*

$$\mathcal{M}_L/\mathcal{G} := \left\{ (\Phi, A) : D^A\Phi = 0, F_A^+ = -\frac{1}{4}\omega^\Phi \right\} /\mathcal{G} \tag{15}$$

is compact.

Sketch of proof. Let (Φ, A) be a solution. From Proposition 2.1 we have an L^∞ bound on Φ. From the second Seiberg-Witten equation we get a bound on F_A^+. From ellipticity of $d_\Gamma^+ + d_\Gamma^*$ we get an L^p bound on F_A for all $p \in (1, \infty)$.

Thus, $\mathcal{M}_L$ is a closed and bounded subset of a metric space and therefore compact. □

A detailed proof is given in [Nic00, Section 2.2.1].

3 The Kähler Case

Proposition 3.1. *The virtual dimension of the moduli space of solutions to the SW-equations on M^4 is given by*

$$\operatorname{vdim} \mathcal{M}_L = \frac{1}{4} c_1^2 - \frac{1}{4}(2\chi + 3\sigma), \tag{16}$$

where c_1 is the first Chern class of L, χ is the Euler characteristic of M, and σ is the signature of M.

A proof is given in [Fri00, p.140].

Proposition 3.2. *On a Kähler manifold:* $\operatorname{vdim} \mathcal{M}_L = 0$.

Proof. On every Kähler manifold we have $\sigma = \frac{1}{3}(c_1^2 - 2c_2)$ and $\chi = c_2$, which implies the claim. □

Remark 3.3. Note that we in fact have $\operatorname{vdim} \mathcal{M}_L = 0$ for any almost-complex structure. A proof of this statement can be found in [Fri00, p.147].

Proposition 3.4. *Let X^4 be Kähler endowed with the canonical* Spin^c *structure constructed in Proposition 1.5. We have the following identifications:*

$$S^+ \simeq \Lambda^{0,0}T^*X \oplus \Lambda^{0,2}T^*X, \qquad S^{\simeq}\Lambda^{0,1}T^*X. \tag{17}$$

$L \simeq K_X$ carries a canonical connection A_0 induced by the Levi-Civita connection, which defines a canonical Dirac operator $D^{A_0} : \Gamma(S^+) \to \Gamma(S^-)$. Under the identifications from line 17 we have:

$$D^{A_0} = \sqrt{2}(\overline{\partial} + \overline{\partial}^*) : \Lambda^{0,0}T^*X \oplus \Lambda^{0,2}T^*X \to \Lambda^{0,1}T^*X. \tag{18}$$

Using this identification as well as $\Lambda^+ \simeq \mathbb{R} \cdot \omega \oplus \Lambda^{0,2}T^*X$, and writing $A \in \mathcal{A}(L)$ as $A = \Gamma + B$, where we consider $B \in \mathcal{A}(\Lambda^{0,0})$ to be a connection on the trivial bundle, we can rewrite the SW-equations as the *Kähler-Seiberg-Witten equations*:

$$\begin{cases} \overline{\partial}_B \alpha & = -\overline{\partial}_B^* \beta \\ F_B^{0,2} & = \overline{\alpha}\beta \\ i\langle F_B^{1,1}, \omega\rangle & = \left(|\beta|^2 - |\alpha|^2\right) - i\langle F_\Gamma^{1,1}, \omega\rangle \end{cases} \tag{KSW}$$

for a triple $(\alpha, \beta, B) \in \Gamma(\Lambda^{0,0}) \times \Gamma(\Lambda^{0,2}) \times \mathcal{A}(\Lambda^{0,0})$.

Proposition 3.5. *Let M^4 be a Kähler manifold with $b^+(M) > 1$ endowed with its canonical* Spin^c *structure. Assume that $\int_M \mathrm{scal}\,\mathrm{dvol}_M \leq 0$. Then the KSW-equations (KSW) have a unique solution.*

Proof. Apply $\overline{\partial}_B$ to the first equation of (KSW) to get:

$$\overline{\partial}_B \overline{\partial}_B^* \beta = -\overline{\partial}_B \overline{\partial}_B \alpha = -F_B^{0,2}\alpha.$$

Plugging the second equation of (KSW) into this expression yields:

$$\overline{\partial}_B \overline{\partial}_B^* \beta + |\alpha|^2 \beta = 0.$$

Multiplying with β and integrating gives:

$$\int_M \left|\overline{\partial}_B^* \beta\right|^2 + |\alpha|^2 |\beta|^2 \,\mathrm{dvol}_M = 0.$$

Thus, there exists a small ball where $\beta \equiv 0$ or $(\alpha, \overline{\partial}_B^* \beta) \equiv (0,0)$. By the unique continuation for elliptic PDE we have that $\beta \equiv 0$ everywhere or $(\alpha, \overline{\partial}_B^* \beta) \equiv (0,0)$ everywhere.

We have $F_B^{0,2} = \overline{a}\beta = 0$. By the Newlander-Nirenberg theorem, B defines a holomorphic structure on the trivial bundle $\underline{\mathbb{C}}$.

Because of the identification $\Lambda^+ = \mathbb{R} \cdot \omega \oplus \Lambda^{0,2}$ and $b^+(M) > 1$ we have a harmonic section of $\Lambda^{0,2}$. Conjugation gives a harmonic section of $\Lambda^{2,0}$, and using $(\mathrm{d} + \mathrm{d}^*)^2 =: \Delta = \Delta^B := \overline{\partial}_B \overline{\partial}_B^* + \overline{\partial}_B^* \overline{\partial}_B$ we have a section ζ of $\Lambda^{2,0}$ such that $\Delta^B \zeta = 0$, and therefore $\overline{\partial}_B \zeta = 0$.

It is a fact for any holomorphic line bundle ξ that $\deg \xi := \langle c_1(\xi), \omega\rangle < 0$ implies that ξ admits no non-zero holomorphic section. Thus:

$$\begin{aligned} 0 &\leq \deg(K_X) \\ &= \langle c_1(K_X), \omega\rangle \\ &= -\langle c_1(K_X^{-1}), \omega\rangle \\ &= -\frac{i}{2\pi}\int_M \langle F_\Gamma, \omega\rangle \,\mathrm{dvol}_M \\ &= \underbrace{\frac{i}{2\pi}\int_M \langle F_B, \omega\rangle \,\mathrm{dvol}_M}_{=0 \text{ because } F_B \text{ exact}} + \frac{1}{2\pi}\int_M |\alpha|^2 - |\beta|^2 \,\mathrm{dvol}_M, \end{aligned}$$

where we used the definition of c_1 via Chern-Weil theory. This implies that $\beta \equiv 0$, $\overline{\partial}_B \alpha = 0$, i.e. α is a holomorphic section with respect to B.

α defines a holomorphic trivialisation of $\underline{\mathbb{C}}$. B is uniquely determined by α and the condition $\overline{\partial}_B \alpha = 0$. Write e^f for the norm of α. The Seiberg-Witten equation becomes:

$$\Delta f + e^{2f} = -iF_\Gamma \cdot \omega = -\frac{1}{8}\,\mathrm{scal}\,. \tag{19}$$

By general PDE theory, this has a unique solution if $-\,\mathrm{scal}$ has non-negative integral, which was assumed. □

4 Symplectic 4-manifolds

(This section is basically a copy of the last paragraph of [Don96])

On a symplectic manifold (M^4, ω) choose a metric and compatible almost compex structure J. As before, we have a canonical Spin^c-structure and much of the analysis can be carried out just as in the Kähler case (cf. [Don96] for an overview of what works the same and what is different in this setting).

Taubes shows that all line bundles for which the suitably perturbed SW-equations have at least one solution are of the form

$$\pm(2\xi - K_X) \tag{20}$$

(i.e. $\xi \otimes \xi \otimes K_X^{-1}$ and $\xi^{-1} \otimes \xi^{-1} \otimes K_X$ respectively), where the Poincaré dual of $\mathrm{PD}(c_1(\xi)) \in H_2(M)$ has non-zero Gromov-invariant: in particular the homology class is represented by a pseudo-holomorphic curve.

Moreover, the "number" of pseudo-holomorphic curves in this homology class equals the Seiberg-Witten invariant. To prove this, Taubes shows that for a solution (α, β, B) the zero set of α can be perturbed to a pseudo-holomorphic curve, thereby setting up a $1:1$-correspondence.

References

[DK90] S.K. Donaldson and P.B. Kronheimer. *The Geometry of Four-manifolds*. Oxford mathematical monographs. Clarendon Press, 1990. 5

[Don96] S. K. Donaldson. The Seiberg-Witten equations and 4-manifold topology. *Bull. Amer. Math. Soc. (N.S.)*, 33(1):45–70, 1996. URL: https://doi.org/10.1090/S0273-0979-96-00625-8, doi:10.1090/S0273-0979-96-00625-8. 8

[Fri00] Thomas Friedrich. *Dirac operators in Riemannian geometry*, volume 25 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2000. Translated from the 1997 German original by Andreas Nestke. URL: https://doi.org/10.1090/gsm/025, doi:10.1090/gsm/025. 2, 3, 4, 6

[FU91] Daniel S. Freed and Karen K. Uhlenbeck. *Instantons and four-manifolds*, volume 1 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, second edition, 1991. URL: https://doi.org/10.1007/978-1-4613-9703-8, doi:10.1007/978-1-4613-9703-8. 5

[KM94] P. B. Kronheimer and T. S. Mrowka. The genus of embedded surfaces in the projective plane. *Math. Res. Lett.*, 1(6):797–808, 1994. URL: https://doi.org/10.4310/MRL.1994.v1.n6.a14, doi:10.4310/MRL.1994.v1.n6.a14. 4

[Nic00] Liviu I. Nicolaescu. *Notes on Seiberg-Witten theory*, volume 28 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2000. URL: https://doi.org/10.1090/gsm/028, doi:10.1090/gsm/028. 6