Ricci flow invariant curvature conditions – Concluding Talk

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BIGW
July 2023

Plan for the talk

- Summary of what we've seen in this series of talks/ Survey of related results.
- $\sec \geqslant 0$, $\sec > 0$ under the Ricci flow in dimension 4
- Further questions.

Introduction/ Recap

- Ricci flow: geometric PDE for evolving a metric $(M^n, \mathrm{g}(t))$.

$$\frac{\partial \mathrm{g}}{\partial t} = -2\,\mathrm{Ric}_{\mathrm{g}(t)}$$
$$\mathrm{g}(0) = \mathrm{g}_0$$

- Expect: lower bounds on curvature preserved under the flow.

- For example, the scalar curvature R evolves as:

$$\frac{\partial R_{\mathrm{g}(t)}}{\partial t} = \Delta_{\mathrm{g}(t)} R_{\mathrm{g}(t)} + 2|\,\mathrm{Ric}_{\mathrm{g}(t)}\,|^2.$$

$\xRightarrow{\text{(scalar) maximum principle}}$ $R \geqslant 0$ is preserved under the Ricci flow.

Ricci flow and positive curvature

- [Hamilton '82] proved that a 3-manifold with $\mathrm{Ric} > 0$ is diffeomorphic to S^3/Γ.
- Along the way, he proved that $\mathrm{Ric} > 0$ and $\sec > 0$ are preserved under RF on a 3-manifold.
- [Hamilton '86] showed that $\mathcal{R} > 0$ is preserved under RF in all dim, and a 4-manifold with $\mathcal{R} > 0$ is diffeomorphic to S^4/Γ.
- [Chen '91] showed that 2-positive curvature operator is RF preserved, and [Böhm–Wilking '08] showed that such manifolds are diffeomorphic to S^n/Γ.
- [Nguyen '07, '10] and [Brendle–Schoen '09] independently showed that PIC is RF preserved, and [Brendle–Schoen '09] used this to prove the differentiable sphere theorem.

Positive curvature cones

- Curvature operator $\mathcal{R}_p : \Lambda^2\mathbb{R}^n \to \Lambda^2\mathbb{R}^n$.
- Positive curvature conditions correspond to cones $\mathcal{C} \subseteq S^2_B\Lambda^2\mathbb{R}^n$.
- PDE $\rightsquigarrow$ ODE

$$\frac{\partial \mathcal{R}}{\partial t} = \Delta\mathcal{R} + Q(\mathcal{R}) \qquad \rightsquigarrow \qquad \frac{d\mathcal{R}}{dt} = Q(\mathcal{R})$$

- Hamilton's tensor maximum principle:
  $\mathcal{C}$ preserved by ODE $\implies$ $\mathcal{C}$ preserved by PDE,
  i.e., the corresponding curvature condition is RF invariant.
- [Wilking '13]: Lie alg. criterion $\rightsquigarrow$ unified proof of all prior RF invariant curvature cones.
- [Bamler – Cabezas-Rivas – Wilking '19]: Almost non-negative curvature conditions.

On the opposite side . . .

- [Maximo, '14] : Ric $\geqslant 0$ is not preserved under the Ricci flow in dim $\geqslant 4$.

- [Ni '04]: example of non-compact M^4 where sec > 0 is not preserved under the RF.

- [Böhm–Wilking '07]: $\exists$ homogeneous metrics of sec > 0 on $M^{12} = \mathrm{Sp}(3)/\mathrm{Sp}(1)\mathrm{Sp}(1)\mathrm{Sp}(1)$ which evolve under the Ricci flow to metrics with mixed Ricci curvature.

  [Cheung–Wallach '15], [Abiev–Nikonorov '16]: similar result on $M^6 = \mathrm{SU}(3)/\mathrm{T}^2$ and $M^{24} = \mathrm{F}_4/\mathrm{Spin}(8)$.

- $\mathcal{C}_{\mathrm{sec}\geqslant 0} \subset S^2_B\Lambda^2\mathbb{R}^n$ is <u>not</u> invariant under the Ricci flow ODE starting in dimension 4.

  [Richard–Seshadri '15]: In any even dimension n, the cone $\mathcal{C}_{scal\geqslant 0}$ is the only RF (ODE) invariant curvature cone that contains $\mathcal{C}_{\mathrm{sec}\geqslant 0}$.

sec $\geqslant 0$ in dimension 4

Theorem 0 [Bettiol – K., 2019]

There exist metrics with sec $\geqslant 0$ on S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ that immediately lose this property (sec $\geqslant 0$) when evolved via the Ricci flow.

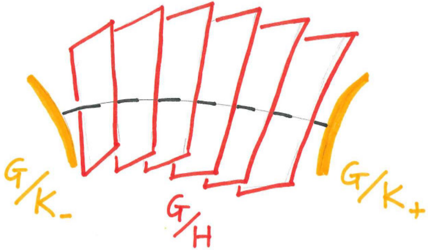
Corollary

sec $\geqslant 0$ *is not preserved under the Ricci flow in dimensions* $\geqslant 4$.

- All of these 4-manifolds admit cohomogeneity one actions by $\mathsf{G} = \mathsf{SO}(3) \rightsquigarrow$ unified proof.

Cohomogeneity one manifolds

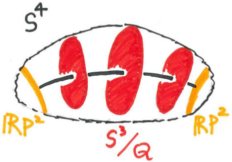
- $G \circlearrowright (M, g)$ such that $M/G = [0, L]$.



- Each orbit is isometric to a homogeneous space, G/H for pre-images of points $\in (0, L)$ (**principal orbits**) and $G/K_\pm$ for pre-images of L and 0 (**singular orbits**).
- $K_\pm/H = S^{\ell_\pm}$.
- M is the union of disk bundles over the singular orbits, $M = \nu_- \bigsqcup \nu_+$, where $\nu_\pm = G \times_{K_\pm} D^{\ell_\pm + 1}$.

A cohomogeneity one action on S^4

$SO(3) \circlearrowright \mathbb{R}^5 = \{A \in M_3(\mathbb{R}) : A = A^T, tr(A) = 0\}$ by conjugation.
Descends to $SO(3) \circlearrowright S^4$.



Invariant metrics

$$\mathrm{g} = dr^2 + \mathrm{g}_r, \quad r \in (0, L)$$

i.e., $$\mathrm{g} = dr^2 + \phi(r)^2\omega_1^2 + \xi(r)^2\omega_2^2 + \psi(r)^2\omega_3^2$$

$\phi(r)$, $\xi(r)$, $\psi(r)$ satisfy **smoothness conditions** at $r = 0$ and $r = L$, determined by the equivariant geometry.

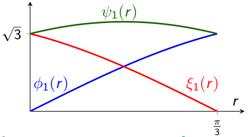


Figure: g_1 = **round metric**

[Grove–Ziller '00]: A cohomogeneity one manifold with singular orbits of codimension 2 admits invariant sec $\geqslant 0$ metrics.

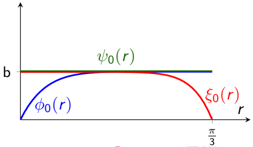
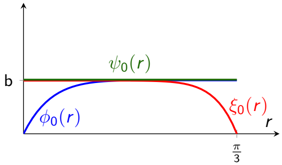


Figure: g_0 = **Grove–Ziller metric**

Proof of Theorem 0

- Let $\mathrm{g}(0) = \mathrm{g}_0 =$ Grove–Ziller metric evolve by the Ricci flow as $\mathrm{g}(t)$.
- Sectional curvature $\sec\left(X_3, \frac{\partial}{\partial r}\right) = -\frac{\psi''}{\psi}$.
- If $\mu(r) = \operatorname{span}\left\{X_3, \frac{\partial}{\partial r}\right\}$, then $\sec_{\mathrm{g}_0}(\mu(r)) = 0$.



- $\mathrm{Isom}(\mathrm{g}(t))$ remains unchanged under the flow, and RF is equivalent to a system of PDEs*

$$\phi_t = \phi_{rr} + \cdots$$
$$\xi_t = \xi_{rr} + \cdots$$
$$\psi_t = \psi_{rr} + \cdots$$

- Using the above coupled PDEs and the smoothness conditions, we compute:

$$\frac{d}{dt}\sec_{\mathrm{g}(t)}(\mu(r))\Big|_{t=0} = \frac{d}{dt}\left(-\frac{\psi''}{\psi}\right)\Big|_{t=0} = \cdots < 0 \quad \text{for } r \text{ small.}$$

□

sec > 0 in dimension 4

Theorem 1 [Bettiol – K., 2023]

There exist metrics with sec > 0 on S^4 and $\mathbb{C}P^2$ that evolve under the Ricci flow to metrics with sectional curvature of mixed sign.

Theorem 2 [Bettiol – K., 2023]

Every Grove–Ziller metric g_0 on S^4 or $\mathbb{C}P^2$ is the limit (in C^∞ topology) of cohomogeneity one metrics g_s with sec > 0.

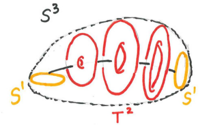
Proof of Theorem 1 (assuming Theorem 2).

- By the continuous dependence of the Ricci flow on initial data, if the initial metric g_s is close to g_0, then $\mathrm{g}_s(t)$ is close to $\mathrm{g}_0(t)$.
- Proof of Theorem 0 $\implies$ $\sec_{\mathrm{g}_0(t)}(\mu) < 0$ for t small. Therefore for s small enough, $\sec_{\mathrm{g}_s(t)}(\mu) < 0$.

□

Perturbing the Grove–Ziller metric to one of sec > 0

Warm-up: $T^2 \circlearrowright S^3 \subset \mathbb{C}^2$



$$g = dr^2 + \phi(r)^2 d\theta_1^2 + \xi(r)^2 d\theta_2^2$$

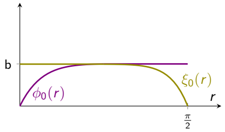


Figure: g_0 = **Grove–Ziller metric**

- Curvature operator $\mathcal{R} = \operatorname{diag}\left(-\frac{\phi''}{\phi}, -\frac{\xi''}{\xi}, -\frac{\phi'\xi'}{\phi\xi}\right)$.
- Recall: in dim 3, $\mathcal{R}$ positive definite $\iff$ sec > 0
- sec $\geqslant 0$ (> 0) iff ϕ, ξ are (strictly) concave and monotone.
- Define g_s by

$$\phi_s = (1-s)\,\phi_0 + s\,\phi_1$$
$$\xi_s = (1-s)\,\xi_0 + s\,\xi_1$$

then $\sec_{g_s} > 0 \ \forall\ s \in (0,1]$.

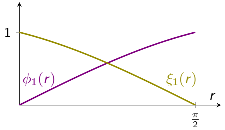


Figure: g_1 = **round metric**

Proof of Theorem 2 (for S^4)

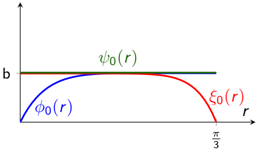


Figure: g_0 = **Grove–Ziller metric**

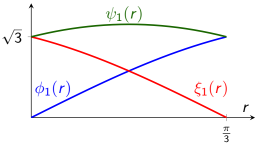


Figure: g_1 = **round metric**

Let g_s be the metric defined by
$$\phi_s = (1-s)\,\phi_0 + s\,\phi_1$$
$$\xi_s = (1-s)\,\xi_0 + s\,\xi_1$$
$$\psi_s = (1-s)\,\psi_0 + s\,\psi_1.$$

CLAIM: $\forall s > 0$ small enough, $\sec_{g_s} > 0$.

In dim 4, $\sec \geqslant 0$ is not equivalent to $\mathcal{R} \geqslant 0$! However, we have:

Thorpe's trick

Let $\mathcal{R} \in S^2_B(\Lambda^2\mathbb{R}^4)$ be an algebraic curvature operator. Then $\mathcal{R}$ has $\sec > 0$ ($\geqslant 0$) iff $\exists\, \tau \in \mathbb{R}$ such that $\mathcal{R} + \tau * > 0$ ($\geqslant 0$).

Proof of Theorem 2 (for S^4) continued

Thorpe's trick

Let $\mathcal{R} \in S^2_B(\Lambda^2\mathbb{R}^4)$ be an algebraic curvature operator. Then $\mathcal{R}$ has sec > 0 ($\geqslant 0$) iff $\exists \tau \in \mathbb{R}$ such that $\mathcal{R} + \tau * > 0$ ($\geqslant 0$).

- $\sec_{g_0} \geqslant 0$, so $\exists\ \tau_0 : [0, L] \to \mathbb{R}$ such that $\mathcal{R}_{g_0} + \tau_0 * \geqslant 0$.

In fact, $\tau_0(r) = -\dfrac{\phi_0'(r)}{2b^2}$, $r \in [0, \frac{L}{2}]$.

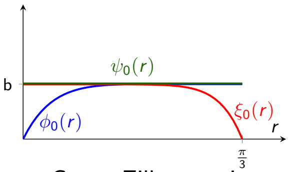


Figure: Grove–Ziller metric on S^4

- Define $\tau_s(r) = \tau_0(r) + O(s)$ so that $\mathcal{R}_{g_s} + \tau_s * > 0$ for s sufficiently small.

□

Remarks and further questions

- Same result in higher dimensions is not an immediate corollary since sec > 0 is not preserved under products, unlike sec $\geqslant 0$.

- A first order perturbation of a sec $\geqslant 0$ metric to attain sec > 0, can be obstructed, e.g. if the initial metric contains a flat totally geodesic embedded torus.

- Odd-dimensional example where sec > 0 is not preserved under Ricci flow?

Thank You!